FORMAL SIMILARITIES BETWEEN UNIDIMENSIONAL FINITE ELEMENTS (1D) USED IN DEFORMABILITY AND DIATHERMANCY ANALYSIS

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INTRODUCTION

The analysis of a system, which takes into account all the operating parameters, is difficult, being almost impossible from practical point of view. That is way, a series of hypotheses concerning the geometry and the constitutive materials of the system are admitted. There are also adopted assumptions for phenomenon causes. This procedure leads to two types of models: the cause model and the system model.

The models can be analytical models or numerical models and they consist in continuous or discrete virtual systems.

The most powerful numerical method for the analysis of both structural deformability and conductive heat transfer is the finite element method.

The present paper approaches the perfect similarity between the problem of axial elastic deformability for a finite element of bar type (1D) and the diathermancy problem for the same finite element.

1. MATRIX EQUILIBRIUM EQUATION FOR OF AN AXIAL ELASTIC DEFORMABLE BAR

The finite element of bar type is the classic one, having two nodes at its ends, at each node the nodal displacement along x axis being defined, d_{xi} (vector) and the corresponding nodal force, f_{xi} (vector). There are defined:

$\{d_x\}^T = \{d_{x1} \quad d_{x2}\}$ - the vector of nodal displacements for axial elastic deformable element,

$\{f_x\}^T = \{f_{x1} \quad f_{x2}\}$ - the vector of nodal forces for the axial elastic deformable element.

For any current section of the element, located at a distance x from the axis origin, Hooke's law can be expressed. It states the relation between the axial stress, $\sigma_x(x)$/axial force $f_x(x)$ and the corresponding displacement gradient $g_{xd_x}(x)$:

$$\{\sigma_x(x)\} = [E_x]\cdot\{\varepsilon_x(x)\} = [E_x]\cdot\left\{\frac{d(d_x(x))}{dx}\right\} \quad (1)$$

or the equivalent relation:

$$\{f_x(x)\} = [E_x A]\cdot\{g_{xd_x}\} \quad (2)$$

The displacement field is:

$$\{d_x(x)\} = \{\alpha_1 + \alpha_2 \cdot x\} = [1 \quad x]\cdot\{\alpha_1 \quad \alpha_2\}^T = \\ = [\phi(x)]\cdot\{\alpha\} \quad (3)$$

For the element nodes can be shortly written that:

$$\begin{Bmatrix} d_x(x_1) = d_{x1} \\ d_x(x_2) = d_{x2} \end{Bmatrix} = \{d_x\} = \begin{Bmatrix} \alpha_1 + \alpha_2 \cdot x_1 \\ \alpha_1 + \alpha_2 \cdot x_2 \end{Bmatrix} = \\ = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}\cdot\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = [A]\{\alpha\} \quad (4)$$

where

$$\{\alpha\} = [A]^{-1}\cdot\{d_x\} \quad (5)$$

and then:

$$\{d_x(x)\} = [\phi(x)]\cdot[A]^{-1}\cdot\{d_x\} = [N(x)]\cdot\{d_x\} = \\ = [N_1(x) \quad N_2(x)]\cdot\{d_{x1} \quad d_{x2}\}^T \quad (6)$$

where $N_i(x)$ represent the shape functions, that for type of element have the form

$$N_1(x) = 1 - \frac{x}{l}, \quad N_2(x) = \frac{x}{l} \quad (7)$$

and their derivatives with respect to x have the expressions:

$$\frac{d}{dx}N_1(x) = -\frac{1}{l},\ \frac{d}{dx}N_2(x) = \frac{1}{l} \quad (8)$$

The displacement gradient function can be related to nodal displacements.

$$\{g_{xd_x}(x)\} = \left\{\frac{d(d_x(x))}{dx}\right\} = \frac{d}{dx}\{d_x(x)\} = = \frac{d}{dx}\{[N(x)]\cdot\{d_x\}\} = \frac{d}{dx}[N(x)]\cdot\{d_x\} = = [B]\cdot\{d_x\} \quad (9)$$

By using Hooke's law it can be stated:

$$\{f_x(x)\} = [E_x A]\cdot\{g_{xT}(x)\} = [E_x A]\cdot[B]\cdot\{d_x\} = = [D]\cdot[B]\cdot\{d_x\} \quad (10)$$

or

$$\{f_x(x)\} = [D]\cdot\left[\frac{d}{dx}N_1(x) \quad \frac{d}{dx}N_2(x)\right]\cdot\{d_x\} = = \left[[D]\cdot\frac{d}{dx}N_1(x) \quad [D]\cdot\frac{d}{dx}N_2(x)\right]\cdot\begin{Bmatrix} d_{x1} \\ d_{x2} \end{Bmatrix} \quad (11)$$

where, by expansion, rearrangement, and substitution of known terms, it is obtained:

$$\begin{bmatrix} \frac{E_x A}{l} & -\frac{E_x A}{l} \\ -\frac{E_x A}{l} & \frac{E_x A}{l} \end{bmatrix}\cdot\begin{Bmatrix} d_{x1} \\ d_{x2} \end{Bmatrix} = \begin{Bmatrix} f_{x1} \\ f_{x2} \end{Bmatrix} \quad (12)$$

that represents the matrix equilibrium equation for an axial elastic deformable bar. It can be shortly expressed as:

$$[k]\cdot\{d_x\} = \{f_x\} \quad (13)$$

where $[k]$ is the stiffness matrix of the element. Equation (13) has been obtained by considering that the axial force at node 1 is a negative one, while the axial force at node 2 is a positive one.

2. MATRIX EQUILIBRIUM EQUATION FOR A DIATHERMIC BAR

The adopted finite element is a uni-dimensional one, having the cross – sectional area *A*. At the nodes, which are provided at the element ends, the temperatures T_i are defined (a scalar quantity) and the corresponding heat flow, Q_{xi} (a vector). There are expressed:

$\{T\}^T = \{T_1 \quad T_2\}$-the vector of nodal temperatures for the diathermic element and

$\{Q_x\}^T = \{Q_{x1} \quad Q_{x2}\}$-the vector of heat flow for the same element.

For any current section of the element, located at a distance x from the origin, Fourier's relation can be written and it states the relation between the heat flux, $q_x(x)$ / heat flow, $Q_x(x)$ and the corresponding temperature gradient, $g_{xT}(x)$.

$$\{q_x(x)\} = [-\lambda_x]\cdot\left\{\frac{d(T(x))}{dx}\right\} \quad (14)$$

where λ_x represents the thermal permeability characteristic.

The equivalent relation is:

$$\{Q_x(x)\} = [-\lambda_x A]\cdot\{g_{xT}(x)\} \quad (15)$$

The function of temperature variation along the element is a first degree function:

$$\{T(x)\} = \{\alpha_1 + \alpha_2\cdot x\} = [1 \quad x]\cdot\{\alpha_1 \quad \alpha_2\}^T = = [\phi(x)]\cdot\{\alpha\} \quad (16)$$

The temperature function can be related to nodal temperatures:

$$\begin{Bmatrix} T(x_1) = T_1 \\ T(x_2) = T_2 \end{Bmatrix} = \{T\} = \begin{Bmatrix} \alpha_1 + \alpha_2\cdot x_1 \\ \alpha_1 + \alpha_2\cdot x_2 \end{Bmatrix} = = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}\cdot\begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = [A]\cdot\{\alpha\} \quad (17)$$

It results that:

$$\{\alpha\} = [A]^{-1}\cdot\{T\} \quad (18)$$

So:

$$\{T(x)\} = [\phi(x)] \cdot [A]^{-1} \cdot \{T\} = [N(x)] \cdot \{T\} = \\ = [N_1(x) \quad N_2(x)] \cdot \{T_1 \quad T_2\}^T \tag{19}$$

where $N_i(x)$ are the shape functions that for an element with two nodes have the following form:

$$N_1(x) = 1 - \frac{x}{l}, \; N_2(x) = \frac{x}{l} \tag{20}$$

and their derivatives with respect to x have the expressions:

$$\frac{d}{dx} N_1(x) = -\frac{1}{l}, \; \frac{d}{dx} N_2(x) = \frac{1}{l} \tag{21}$$

The function of temperature gradient is expressed in terms of nodal temperatures:

$$\{g_{xT}(x)\} = \left\{\frac{d(T(x))}{dx}\right\} = \frac{d}{dx}\{T(x)\} = \\ = \frac{d}{dx}\{[N(x)] \cdot \{T\}\} = \frac{d}{dx}[N(x)] \cdot \{T\} = [B] \cdot \{T\} \tag{22}$$

By using Fourier's relation, it is obtained:

$$\{Q_x(x)\} = [-\lambda_x A] \cdot \{g_{xT}(x)\} = \\ = [-\lambda_x A] \cdot [B] \cdot \{T\} = [D] \cdot [B] \cdot \{T\} \tag{23}$$

or:

$$\{Q_x(x)\} = [D] \cdot \left[\frac{d}{dx} N_1(x) \quad \frac{d}{dx} N_2(x)\right] \cdot \{T\} = \\ = \left[[D] \cdot \frac{d}{dx} N_1(x) \quad [D] \cdot \frac{d}{dx} N_2(x)\right] \cdot \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \tag{24}$$

which, by expansion, rearrangement and substitution of known terms becomes:

$$\begin{bmatrix} \frac{\lambda_x A}{l} & -\frac{\lambda_x A}{l} \\ -\frac{\lambda_x A}{l} & \frac{\lambda_x A}{l} \end{bmatrix} \cdot \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} Q_{x1} \\ Q_{x2} \end{Bmatrix} \tag{25}$$

that represents the matrix equilibrium equation of the diathermic bar, expressed in terms of temperatures, and which can be shortly expressed:

$$[\lambda] \cdot \{T\} = \{Q_x\} \tag{26}$$

where $[\lambda]$ is the permeability matrix of the diathermic bar.

Equation (26) has been obtained by considering that the heat flow which enters node 1 is positive, while the heat flow which exits node 2 is negative.

3. CONCLUSIONS

By comparing the matrix equilibrium equation (12), stated for the axial elastic deformability case, with equation (25), valid for diathermancy case, it can be noticed that they have the same shape. In the second mentioned equation, the longitudinal modulus of elasticity E_x that occurs in the first equation, was substituted by the thermal permeability characteristic, λ_x.

Taking into account that the two equilibrium equations are identical from a formal point of view, it can be concluded that these procedures concerning the elastic deformability analysis can be also applied for heat transfer problems, at least for the stationary case.

Such formal similarities between different phenomena offer to the analysts the possibility to perform analogies, but also to use and / or extend design methods of apparently different engineering problems.

The existing soft for deformability analysis could be easily adapted for heat transfer problems, but also for other problems, as fluid flow or pressure.

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