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Dedicated to Professor Alexandru Şubă on the occasion of his 70^{th} birthday

Quartic differential systems with a non-degenerate monodromic critical point and multiple line at infinity

Alexandru Şubă and Olga Vacaraş

Abstract. The quartic differential systems with a non-degenerate monodromic critical point and non-degenerate infinity are considered. We show that in this family the maximal multiplicity of the line at infinity is seven. Modulo the affine transformation and time rescaling the classes of systems with the line of infinity of multiplicity two, three, ..., seven are determined. In the cases when the quartic systems have the line at infinity of maximal multiplicity the problem of the center is solved.

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Keywords: quartic differential system, multiple invariant line, monodromic critical point.

Sistemele diferenţiale cuartice ce au punct critic monodromic nedegenerat şi linia de la infinit multiplă

Rezumat. În această lucrare sunt examinate sistemele diferenţiale cuartice cu un punct critic monodromic nedegenerat şi infinitul nedegenerat. Se arată că în această familie de sisteme multiplicitatea maximală a dreptei de la infinit este egală cu şapte. Cu exactitatea unei transformări afine de coordonate şi rescalarea timpului sunt determinate clasele de sisteme cu dreapta de la infinit de multiplicitatea doi, trei, ..., şapte. În cazurile când sistemele cuartice au linia de la infinit de multiplicitate maximală problema centrului este rezolvată.

Cuvinte-cheie: sistem diferenţial cuartic, dreaptă invariantă multiplă, punct critic monodromic.

1. Introduction

We consider real polynomial differential systems

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1}$$

where $\dot{x} = dx/dt,\ \dot{y} = dy/dt$.

Let $n = max\{deg(p),\ deg(q)\}$. If $n = 2$ (respectively, $n = 3,\ n = 4$), then system (1) is called quadratic (respectively, cubic, quartic). Via an affine transformation of coordinates and time rescaling each non-degenerate quartic system with a non-degenerate infinity and a center-focus critical point, i.e. a critical point with pure imaginary eigenvalues, can be written in the form

$$\begin{cases} \dot{x} = y + p_2(x,y) + p_3(x,y) + p_4(x,y) \equiv p(x,y), \\ \dot{y} = -(x + q_2(x,y) + q_3(x,y) + q_4(x,y)) \equiv q(x,y), \\ \gcd(p,q) = 1,\ yp_4(x,y) + xq_4(x,y) \not\equiv 0, \end{cases} \tag{2}$$

where $p_i(x,y) = \sum_{j=0}^{i} a_{i-j,j}x^{i-j}y^j$, $q_i(x,y) = \sum_{j=0}^{i} b_{i-j,j}x^{i-j}y^j$, $i = 2,3,4$ are homogeneous polynomials in x and y of degree i with real coefficients.

The eigenvalues λ_1, λ_2 of a critical point $(0,0)$ of system (2) are complex, $\lambda_1\lambda_2 \neq 0$, $\lambda_2 = \overline{\lambda_1}$, and therefore $(0,0)$ is a non-degenerate monodromic critical point.

Remark 1.1. *Via a transformation of the form*

$$x \to \omega(x\cos\varphi - y\sin\varphi)),\ y \to \omega(x\sin\varphi + y\cos\varphi)), \omega \neq 0,$$

and time rescaling we can make in (2)

$$b_{40} = 1. \tag{3}$$

The homogeneous system associated to the quartic system (2) look as

$$\begin{cases} \dot{x} = yZ^3 + p_2(x,y)Z^2 + p_3(x,y)Z + p_4(x,y) \equiv P(x,y,Z), \\ \dot{y} = -(xZ^3 + q_2(x,y)Z^2 + q_3(x,y)Z + q_4(x,y)) \equiv Q(x,y,Z). \end{cases} \tag{4}$$

Denote $\mathbb{X}_\infty = P(x,y,Z)\frac{\partial}{\partial x} + Q(x,y,Z)\frac{\partial}{\partial y}$ and $\mathbb{E}_\infty = P \cdot \mathbb{X}_\infty(Q) - Q \cdot \mathbb{X}_\infty(P)$.

The polynomial $\mathbb{E}_\infty$ has the form

$$\begin{aligned} \mathbb{E}_\infty = {} & A_2(x,y) + A_3(x,y)Z + A_4(x,y)Z^2 + A_5(x,y)Z^3 \\ & + A_6(x,y)Z^4 + A_7(x,y)Z^5 + A_8(x,y)Z^6 + A_9(x,y)Z^7 \\ & + A_{10}(x,y)Z^8 + A_{11}(x,y)Z^9, \end{aligned} \tag{5}$$

where $A_i(x,y)$, $i = 2, ..., 11$, are polynomials in x and y.

We say that for system (2) the line at infinity $Z = 0$ has *multiplicity* ν if $A_2(x,y) \equiv 0, ..., A_\nu(x,y) \equiv 0$, $A_{\nu+1}(x,y) \not\equiv 0$, i.e. $\nu - 1$ is the greatest positive integer such that $Z^{\nu-1}$ divides $\mathbb{E}_\infty$. If $A_2(x,y) \not\equiv 0$, then we say that $Z = 0$ has multiplicity one. Denote by $m(Z)$ the multiplicity of the line at infinity $Z = 0$.

About the notion of multiplicity of an invariant algebraic line and, in particular, of the line at infinity, we recommend the work [1] to the readers.

The quadratic (respectively, cubic; quartic) differential systems with multiple line at infinity was examined in [2] (respectively, [3] – [9]; [10]).

In this paper we will show that the maximal multiplicity of the line at infinity for quartic systems (2) is seven. Moreover, we determine the classes of systems $\{(2), (3)\}$ having the line at infinity of multiplicity two, three, ..., seven.

2. Quartic systems $\{(2), (3)\}$ with the line at infinity $Z = 0$ of multiplicity $m(Z) = 2, 3, 4, 5, 6$

2.1. Systems $\{(2), (3)\}$ with $m(Z) \geq 2$.

The multiplicity of the line at infinity is at least two if the identity $A_2(x, y) \equiv 0$ holds. The polynomial $A_2(x, y)$ looks as $A_2(x, y) = -A_{21}(x, y)A_{22}(x, y)$, where $A_{21}(x, y) = x^5 + (a_{40} + b_{31})x^4y + (a_{31} + b_{22})x^3y^2 + (a_{22} + b_{13})x^2y^3 + (a_{13} + b_{04})xy^4 + a_{04}y^5$, i.e. $A_{21}(x, y) = yp_4(x, y) + xq_4(x, y)$,

and

$A_{22}(x, y) = (a_{31} - a_{40}b_{31})x^6 + 2(a_{22} - a_{40}b_{22})x^5y + (3a_{13} - 3a_{40}b_{13} - a_{31}b_{22} + a_{22}b_{31})x^4y^2 + 2(2a_{04} - 2a_{40}b_{04} - a_{31}b_{13} + a_{13}b_{31})x^3y^3 - (3a_{31}b_{04} + a_{22}b_{13} - a_{13}b_{22} - 3a_{04}b_{31})x^2y^4 - 2(a_{22}b_{04} - a_{04}b_{22})xy^5 - (a_{13}b_{04} - a_{04}b_{13})y^6$.

As $A_{21} \not\equiv 0$, we require A_{22} to be identically equal to zero. Solving the identity $A_{22} \equiv 0$ we obtain the following result:

Lemma 2.1. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity at least two if and only if the coefficients of* $\{(2), (3)\}$ *verify the following conditions:*

$$a_{31} = a_{40}b_{31},\ a_{22} = a_{40}b_{22},\ a_{13} = a_{40}b_{13},\ a_{04} = a_{40}b_{04}. \tag{6}$$

2.2. Systems $\{(2), (3)\}$ with $m(Z) \geq 3$.

The multiplicity $m(Z_\infty)$ of the line at infinity is at least three if $\{A_2(x, y) \equiv 0,\ A_3(x, y) \equiv 0.\}$ In the conditions of Lemma 2.1 the identity $A_3(x, y) \equiv 0$ leads us to the following two series of conditions:

$$a_{30} = a_{40}b_{30},\ a_{21} = a_{40}b_{21},\ a_{12} = a_{40}b_{12},\ a_{03} = a_{40}b_{03}; \tag{7}$$

$$\begin{array}{l} a_{03} = a_{40}b_{03} - a_{30}a_{40}^3 + a_{30}b_{13} - a_{30}a_{40}b_{22} + a_{40}^4b_{30} - a_{40}b_{13}b_{30} \\ +a_{40}^2b_{22}b_{30} + a_{30}a_{40}^2b_{31} - a_{40}^3b_{30}b_{31},\ a_{12} = a_{30}a_{40}^2 + a_{40}b_{12} + a_{30}b_{22} \\ -a_{40}^3b_{30} - a_{40}b_{22}b_{30} - a_{30}a_{40}b_{31} + a_{40}^2b_{30}b_{31},\ a_{21} = a_{40}b_{21} - a_{30}a_{40} \\ +a_{40}^2b_{30} + a_{30}b_{31} - a_{40}b_{30}b_{31},\ b_{04} = a_{40}(b_{13} - a_{40}^3 - a_{40}b_{22} + a_{40}^2b_{31}). \end{array} \tag{8}$$

Lemma 2.2. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity at least three if and only if the coefficients of* $\{(2), (3)\}$ *verify one of the following two sets of conditions: 1)* $\{(6), (7)\}$*; 2)* $\{(6), (8)\}$.

2.3. Systems $\{(2), (3)\}$ with $m(Z) \geq 4$.

In each of the sets of equalities 1) and 2) of Lemma 2.2, the identity $A_4(x, y) \equiv 0$ yields the following series of conditions, respectively:

1) $\{(6), (7)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$a_{20} = a_{40}b_{20},\ a_{11} = a_{40}b_{11},\ a_{02} = a_{40}b_{02}; \tag{9}$$

$$\begin{aligned} &a_{11} = -2a_{20}a_{40} + a_{40}b_{11} + 2a_{40}^2b_{20} + a_{20}b_{31} - a_{40}b_{20}b_{31},\\ &a_{02} = 3a_{20}a_{40}^2 + a_{40}b_{02} - 3a_{40}^3b_{20} + a_{20}b_{22} - a_{40}b_{20}b_{22}\\ &-2a_{20}a_{40}b_{31} + 2a_{40}^2b_{20}b_{31},\ b_{13} = a_{40}(4a_{40}^2 + 2b_{22} - 3a_{40}b_{31}),\\ &b_{04} = a_{40}^2(3a_{40}^2 + b_{22} - 2a_{40}b_{31}),\ \ a_{20} \neq a_{40}b_{20}; \end{aligned} \tag{10}$$

2) $\{(6), (8)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{aligned} &a_{11} = -a_{30}^2 - 2a_{20}a_{40} + a_{40}b_{11} + 2a_{40}^2b_{20} + a_{30}b_{21} + 3a_{30}a_{40}b_{30}\\ &-a_{40}b_{21}b_{30} - 2a_{40}^2b_{30}^2 + a_{20}b_{31} - a_{40}b_{20}b_{31} - a_{30}b_{30}b_{31} + a_{40}b_{30}^2b_{31},\\ &a_{02} = 3a_{30}^2a_{40} + 3a_{20}a_{40}^2 + a_{40}b_{02} + a_{30}b_{12} - 3a_{40}^3b_{20} - a_{30}a_{40}b_{21}\\ &\quad +a_{20}b_{22} - a_{40}b_{20}b_{22} - 8a_{30}a_{40}^2b_{30} - a_{40}b_{12}b_{30} + a_{40}^2b_{21}b_{30}\\ &\quad -a_{30}b_{22}b_{30} + 5a_{40}^3b_{30}^2 + a_{40}b_{22}b_{30}^2 - a_{30}^2b_{31} - 2a_{20}a_{40}b_{31}\\ &\quad +2a_{40}^2b_{20}b_{31} + 4a_{30}a_{40}b_{30}b_{31} - 3a_{40}^2b_{30}^2b_{31},\\ &b_{13} = a_{40}(4a_{40}^2 + 2b_{22} - 3a_{40}b_{31}), \qquad b_{03} = 6a_{30}a_{40}^2 + a_{40}b_{12}\\ &-a_{40}^2b_{21} + a_{30}b_{22} - 5a_{40}^3b_{30} - a_{40}b_{22}b_{30} - 3a_{30}a_{40}b_{31} + 3a_{40}^2b_{30}b_{31}; \end{aligned} \tag{11}$$

$$a_{20} = a_{40}b_{20},\ a_{11} = a_{40}b_{11},\ a_{02} = a_{40}b_{02},\ a_{30} = a_{40}b_{30}; \tag{12}$$

$$\begin{aligned} &a_{11} = -2a_{20}a_{40} + a_{40}b_{11} + 2a_{40}^2b_{20} + a_{20}b_{31} - a_{40}b_{20}b_{31},\\ &a_{02} = 3a_{20}a_{40}^2 + a_{40}b_{02} - 3a_{40}^3b_{20} + a_{20}b_{22} - a_{40}b_{20}b_{22} - 2a_{20}a_{40}b_{31}\\ &+2a_{40}^2b_{20}b_{31}, \quad a_{30} = a_{40}b_{30}, \quad b_{13} = a_{40}(4a_{40}^2 + 2b_{22} - 3a_{40}b_{31}). \end{aligned} \tag{13}$$

It is easy to see that the set of conditions $\{(6), (8), (12)\}$ is a particular case for the set of conditions $\{(6), (7), (9)\}$. The conditions $\{(6), (7), (10)\}$ and $\{(6), (8), (13)\}$ are the same.

Lemma 2.3. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity at least four if and only if the coefficients of* $\{(2), (3)\}$ *verify one of the following three sets of conditions: 1)* $\{(6), (7), (9)\}$*; 2)* $\{(6), (7), (10)\}$*; 3)* $\{(6), (8), (11)\}$.

2.4. Systems $\{(2), (3)\}$ with $m(Z) \geq 5$.

In the conditions of Lemma 2.3 we solve the identity $A_5(x,y) \equiv 0$. We have, respectively:

1) $\{(6), (7), (9)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$\begin{aligned} &b_{31} = (3a_{40}^2 - 1)/a_{40}, \quad b_{22} = 3(a_{40}^2 - 1), \quad b_{13} = a_{40}(a_{40}^2 - 3),\\ &b_{04} = -a_{40}^2, \quad a_{40} \neq 0. \end{aligned} \tag{14}$$

2) $\{(6), (7), (10)\} \Rightarrow A_5(x, y) \equiv 0 \Rightarrow$

$$\begin{aligned} b_{21} &= (1 - 3a_{40}^2 - a_{20}a_{40}b_{30} + a_{40}^2 b_{20}b_{30} + a_{40}b_{31} + a_{20}b_{30}b_{31} \\ &\quad -a_{40}b_{20}b_{30}b_{31})/(a_{20} - a_{40}b_{20}), \qquad b_{12} = (2a_{40} + a_{40}b_{22} \\ &\quad +a_{20}a_{40}^2 b_{30} - a_{40}^3 b_{20}b_{30} + a_{20}b_{22}b_{30} - a_{40}b_{20}b_{22}b_{30} - a_{40}^2 b_{31} \\ &\quad -a_{20}a_{40}b_{30}b_{31} + a_{40}^2 b_{20}b_{30}b_{31})/(a_{20} - a_{40}b_{20}), \\ b_{03} &= a_{40}(a_{40} + 3a_{40}^3 + a_{40}b_{22} + 3a_{20}a_{40}^2 b_{30} - 3a_{40}^3 b_{20}b_{30} \\ &\quad +a_{20}b_{22}b_{30} - a_{40}b_{20}b_{22}b_{30} - 2a_{40}^2 b_{31} - 2a_{20}a_{40}b_{30}b_{31} \\ &\quad +2a_{40}^2 b_{20}b_{30}b_{31})/(a_{20} - a_{40}b_{20}). \end{aligned} \tag{15}$$

3) $\{(6), (8), (11)\} \Rightarrow A_5(x, y) \equiv 0 \Rightarrow$

$$\begin{aligned} b_{22} &= 3a_{40}(b_{31} - 2a_{40}), \\ b_{11} &= (1 + 3a_{20}a_{30} - 3a_{40}^2 - 5a_{30}a_{40}b_{20} - a_{20}b_{21} + a_{40}b_{20}b_{21} \\ &\quad -2a_{30}^2 b_{30} - 4a_{20}a_{40}b_{30} + 6a_{40}^2 b_{20}b_{30} + a_{30}b_{21}b_{30} + 5a_{30}a_{40}b_{30}^2 \\ &\quad -a_{40}b_{21}b_{30}^2 - 3a_{40}^2 b_{30}^3 + a_{40}b_{31} + a_{30}b_{20}b_{31} + a_{20}b_{30}b_{31} \\ &\quad -2a_{40}b_{20}b_{30}b_{31} - a_{30}b_{30}^2 b_{31} + a_{40}b_{30}^3 b_{31})/(a_{30} - a_{40}b_{30}), \\ b_{12} &= -8a_{30}a_{40} + 2a_{40}b_{21} + 5a_{40}^2 b_{30} + 2a_{30}b_{31} - 2a_{40}b_{30}b_{31}, \\ b_{02} &= (-2a_{30}^3 + a_{40} - a_{20}a_{30}a_{40} - 3a_{40}^3 - 2a_{30}a_{40}^2 b_{20} + a_{30}^2 b_{21} \\ &\quad -a_{20}a_{40}b_{21} + a_{40}^2 b_{20}b_{21} + 5a_{30}^2 a_{40}b_{30} + 3a_{40}^3 b_{20}b_{30} \\ &\quad -a_{30}a_{40}b_{21}b_{30} - 3a_{30}a_{40}^2 b_{30}^2 + a_{20}a_{30}b_{31} + a_{40}^2 b_{31} - a_{30}^2 b_{30}b_{31} \\ &\quad -a_{40}^2 b_{20}b_{30}b_{31} + a_{30}a_{40}b_{30}^2 b_{31})/(a_{30} - a_{40}b_{30}), \quad a_{30} \neq a_{40}b_{30}. \end{aligned} \tag{16}$$

Lemma 2.4. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity at least five if and only if the coefficients of* $\{(2), (3)\}$ *verify one of the following three sets of conditions: 1)* $\{(6), (7), (9), (14)\}$*; 2)* $\{(6), (7), (10), (15)\}$*; 3)* $\{(6), (8), (11), (16)\}$.

2.5. Systems $\{(2), (3)\}$ with $m(Z) \geq 6$.

In each of the conditions 1) - 3) of Lemma 2.4 we solve the identity $A_6(x, y) \equiv 0$. We obtain the following results:

1) $\{(6), (7), (9), (14)\} \Rightarrow A_6(x, y) \equiv 0 \Rightarrow$

$$b_{03} = -a_{40}b_{30}, \; b_{12} = (a_{40}^2 - 2)b_{30}, \; b_{21} = (2a_{40}^2 - 1)b_{30}/a_{40}. \tag{17}$$

2) $\{(6), (7), (10), (15)\} \Rightarrow A_6(x, y) \equiv 0 \Rightarrow$

$$\begin{aligned} b_{02} &= (a_{40}^2 - 3a_{40}^4 + a_{40}^3 b_{31} + a_{40}b_{30}\alpha - 3a_{40}^3 b_{30}\alpha + a_{40}^2 b_{30}b_{31}\alpha - \\ &\qquad -3a_{40}^2 b_{20}\alpha^2 + a_{40}b_{20}b_{31}\alpha^2 - 6a_{40}\alpha^3 + 2b_{31}\alpha^3)/\alpha^2, \\ b_{11} &= (a_{40} - 3a_{40}^3 + a_{40}^2 b_{31} + b_{30}\alpha - 3a_{40}^2 b_{30}\alpha + a_{40}b_{30}b_{31}\alpha - \\ &\qquad -2a_{40}b_{20}\alpha^2 + b_{20}b_{31}\alpha^2 + 2\alpha^3)/\alpha^2, \\ b_{22} &= 3a_{40}(b_{31} - 2a_{40}), \quad \alpha = a_{20} - a_{40}b_{20}, \quad \alpha \neq 0. \end{aligned} \tag{18}$$

3) $\{(6), (8), (11), (16)\} \Rightarrow A_6(x, y) \equiv 0 \Rightarrow$

$$\begin{aligned} a_{20} = (1 + a_{40}^2 + 2a_{30}a_{40}b_{20} + a_{30}^2 b_{30} - 2a_{40}^2 b_{20}b_{30} - 2a_{30}a_{40}b_{30}^2 + \\ + a_{40}^2 b_{30}^3)/(2(a_{30} - a_{40}b_{30})), \quad b_{21} = 3a_{30}, \; b_{31} = 4a_{40}. \end{aligned} \tag{19}$$

Lemma 2.5. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity at least six if and only if the coefficients of* $\{(2), (3)\}$ *verify one of the following sets of conditions: 1)* $\{(6), (7), (9), (14), (17)\}$*; 2)* $\{(6), (7), (10), (15), (18)\}$*; 3)* $\{(6), (8), (11), (16), (19)\}$.

3. Quartic systems $\{(2), (3)\}$ with the line at infinity $Z = 0$ of multiplicity seven

To obtain the quartic systems $\{(2), (3)\}$, which have the line at infinity of multiplicity seven, we solve the identity $A_7(x, y) \equiv 0$ in each of the series of conditions 1) – 3) of Lemma 2.5.

1) $\{(6), (7), (9), (14), (17)\} \Rightarrow A_7(x, y) \equiv 0 \Rightarrow$

$$b_{11} = (a_{40}^2 - 1)b_{20}/a_{40}, \qquad b_{02} = -b_{20}. \tag{20}$$

In this case the identity $A_8(x, y) = -(a_{40}x - y)(x + a_{40}y)^2((1 + 3a_{40}^2)x^2 - 4a_{40}xy + (3 + a_{40}^2)y^2)/a_{40} \not\equiv 0$, therefore the multiplicity of the line at infinity is exactly seven.

2) $\{(6), (7), (10), (15), (18)\} \Rightarrow A_7(x, y) \equiv 0 \Rightarrow$

$$\begin{aligned} b_{20} = (2a_{40} + 2a_{40}^3 - 2\alpha^3 + 2a_{40}^2\beta + 2a_{40}^4\beta - 3a_{40}\alpha^3\beta + \\ + \alpha^3\beta^2)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)), \quad \beta = b_{31} - 3a_{40}, \quad \beta \neq 0, \\ b_{30} = (a_{40}\beta - 3a_{40}^2 - 2)/(\alpha(a_{40} - \beta)), \quad (a_{40} - \beta)(1 + a_{40}\beta) \neq 0. \end{aligned} \tag{21}$$

Under these conditions the identity $A_8(x, y) \equiv 0$ does not give us real solutions, therefore the multiplicity of the line at infinity is exactly seven.

3) $\{(6), (8), (11), (16), (19)\}$. In this case the identity $A_7(x, y) \equiv 0$ does not give us real solutions, therefore the multiplicity of the line at infinity is exactly six.

Lemma 3.1. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity seven if and only if the coefficients of* $\{(2), (3)\}$ *verify one of the following sets of conditions:*
1) $\{(6), (7), (9), (14), (17), (20)\}$*; 2)* $\{(6), (7), (10), (15), (18), (21)\}$.

In this way we prove the statement of the following theorem.

Theorem 3.1. *In the class of quartic differential systems with a non-degenerate monodromic critical point and non-degenerate infinity the maximal multiplicity of the line at infinity is seven.*

4. Solution of the center problem for quartic systems $\{(2), (3)\}$ with the line at infinity of maximal multiplicity

Let $F(x,y) = x^2+y^2+F_3(x,y)+F_4(x,y)+\cdots+F_n(x,y)+\cdots$, where $F_k(x,y) = \sum_{i+j=k} f_{ij}x^iy^j$, $f_{0j}=0$ if j is even, be a function such that

$$\frac{\partial F}{\partial x}p(x,y) + \frac{\partial F}{\partial y}q(x,y) \equiv \sum_{j=1}^{\infty} L_j(x^2+y^2)^{j+1}. \tag{22}$$

In (22) L_j are polynomials in coefficients of (2) and are called Lyapunov quantities. The critical point $(0,0)$ is a center for system (2) if and only if $L_j = 0,\ j = 1,2,3,\dots$.

Using the identity (22) we calculate the first three Lyapunov quantities, i.e. we solve in f_{ij} and L_k the following systems of identities:

$$\begin{aligned}
&2xp_2 + yF'_{3x} - 2yq_2 - xF'_{3y} \equiv 0,\\
&2xp_3 + F'_{3x}p_2 + yF'_{4x} - 2yq_3 - F'_{3y}q_2 - xF'_{4y} \equiv L_1(x^2+y^2)^2,\\
&2xp_4 + F'_{3x}p_3 + F'_{4x}p_2 + yF'_{5x} - 2yq_4 - F'_{3y}q_3 - F'_{4y}q_2 - F'_{5y}x \equiv 0,\\
&F'_{3x}p_4 + F'_{4x}p_3 + F'_{5x}p_2 + F'_{6x}y - F'_{3y}q_4 - F'_{4y}q_3 - F'_{5y}q_2 - F'_{6y}x \equiv L_2(x^2+y^2)^3,\\
&F'_{4x}p_4 + F'_{5x}p_3 + F'_{6x}p_2 + F'_{7x}y - F'_{4y}q_4 - F'_{5y}q_3 - F'_{6y}q_2 - F'_{7y}x \equiv 0,\\
&F'_{5x}p_4 + F'_{6x}p_3 + F'_{7x}p_2 + F'_{8x}y - F'_{5y}q_4 - F'_{6y}q_3 - F'_{7y}q_2 - F'_{8y}x \equiv L_3(x^2+y^2)^4.
\end{aligned}$$

The first Lyapunov quantity looks as

$$L_1 = (a_{12} - a_{02}a_{11} - a_{11}a_{20} + 3a_{30} + 2a_{02}b_{02} - 3b_{03} + b_{02}b_{11} - 2a_{20}b_{20} + b_{11}b_{20} - b_{21})/4.$$

The quantities L_2 and L_3 are very cumbersome and cannot be brought here.

In the following we will solve the center problem for the system (2) under the conditions 1) and 2) of Lemma 3.1, i.e. when the line at infinity is of the maximal multiplicity.

The conditins 1) of Lemma 3.1 are

$$\begin{aligned}
&a_{20} = a_{40}b_{20}, a_{11} = a_{40}b_{11}, a_{02} = a_{40}b_{02}, b_{11} = (a_{40}^2-1)b_{20}/a_{40},\\
&b_{02} = -b_{20}, a_{30} = a_{40}b_{30}, a_{21} = a_{40}b_{21}, a_{12} = a_{40}b_{12}, a_{03} = a_{40}b_{03},\\
&b_{21} = (2a_{40}^2-1)b_{30}/a_{40}, b_{12} = (a_{40}^2-2)b_{30}, b_{03} = -a_{40}b_{30}, a_{31} = 3a_{40}^2-1,\\
&a_{22} = 3a_{40}(a_{40}^2-1), a_{13} = a_{40}^2(a_{40}^2-3), a_{04} = -a_{40}^3, b_{40} = 1,\\
&b_{31} = (3a_{40}^2-1)/a_{40}, b_{22} = 3(a_{40}^2-1), b_{13} = a_{40}(a_{40}^2-3), b_{04} = -a_{40}^2.
\end{aligned} \tag{23}$$

The first Lyapunov quantity calculated for system $\{(2), (23)\}$ is $L_1 = (1+a_{40}^2)^2b_{30}/(4a_{40})$ and $L_1 = 0$ gives us $b_{30} = 0$. The transformation $X = a_{40}x - y$, $Y = x + a_{40}y$ reduces system $\{(2), (23), b_{30} = 0\}$ to the following system

$$\begin{aligned}
&\dot{X} = Y(a_{40} + b_{20}X + a_{40}^2b_{20}X + XY^2 + a_{40}^2XY^2))/a_{40},\\
&\dot{Y} = -X.
\end{aligned} \tag{24}$$

For system (24) the straight line $X = 0$ is an axis of symmetry. Therefore, the origin $(X, Y) = (0, 0)$ (respectively, $(x, y) = (0, 0)$) is for system (24) (respectively, $\{(2), (23), b_{30} = 0\}$) the critical point of a center type.

The exponential factors. Let $h, g \in \mathbb{C}[x, y]$ be relatively prime in the ring $\mathbb{C}[x, y]$. The function $\Phi = \exp(g/h)$ is called an *exponential factor* of system (2) if for some polynomial $K \in \mathbb{C}[x, y]$ of degree at most three it satisfies the identity

$$\frac{\partial \Phi}{\partial x} p(x, y) + \frac{\partial \Phi}{\partial y} q(x, y) \equiv \Phi \cdot K(x, y).$$

We remark that the system $\{(2), (23)\}$ has the following six exponential factors

$$\begin{aligned}
&\Phi_1 = \exp(x + a_{40}y),\\
&\Phi_2 = \exp((x + a_{40}y)^2),\\
&\Phi_3 = \exp((x + a_{40}y)^3),\\
&\Phi_4 = \exp(-4a_{40}y + (x + a_{40}y)^4),\\
&\Phi_5 = \exp(-5a_{40}y(x + a_{40}y) + (x + a_{40}y)^5),\\
&\Phi_6 = \exp(6a_{40}b_{20}y - 6a_{40}y(x + a_{40}y)^2 + (x + a_{40}y)^6).
\end{aligned}$$

The conditions 2) of Lemma 3.1 are

$$\begin{aligned}
a_{20} = &-(-2a_{40}^2 - 2a_{40}^4 + a_{40}\alpha^3 - 2a_{40}^3\beta - 2a_{40}^5\beta + \alpha^3\beta +\\
&2a_{40}^2\alpha^3\beta)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
a_{11} = &(-2a_{40} + 2a_{40}^5 + a_{40}^2\alpha^3 - 2a_{40}^2\beta + 2a_{40}^6\beta - 4a_{40}\alpha^3\beta - \alpha^3\beta^2 -\\
&4a_{40}^2\alpha^3\beta^2)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
a_{02} = &-(a_{40}(2a_{40} + 2a_{40}^3 + 2a_{40}^2\beta + 2a_{40}^4\beta - a_{40}\alpha^3\beta + 3\alpha^3\beta^2 +\\
&2a_{40}\alpha^3\beta^3))/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
b_{20} = &-(-2a_{40} - 2a_{40}^3 + 2\alpha^3 - 2a_{40}^2\beta - 2a_{40}^4\beta + 3a_{40}\alpha^3\beta -\\
&\alpha^3\beta^2)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
b_{11} = &-(2 - 2a_{40}^4 + 2a_{40}\beta - 2a_{40}^5\beta + 4\alpha^3\beta + a_{40}^2\alpha^3\beta + 4a_{40}\alpha^3\beta^2 -\\
&\alpha^3\beta^3)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
b_{02} = &-(2a_{40} + 2a_{40}^3 + 2a_{40}^2\beta + 2a_{40}^4\beta + 2\alpha^3\beta^2 + a_{40}^2\alpha^3\beta^2 +\\
&a_{40}\alpha^3\beta^3)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
a_{30} = &-(a_{40}(2 + 3a_{40}^2 - a_{40}\beta))/(\alpha(a_{40} - \beta)),\\
a_{21} = &-(3a_{40}(a_{40} + 2a_{40}^3 + \beta))/(\alpha(a_{40} - \beta)),\\
a_{12} = &-(3a_{40}^2(a_{40}^3 + 2\beta + a_{40}^2\beta))/(\alpha(a_{40} - \beta)),\\
a_{03} = &-(a_{40}^3(-a_{40} + 3\beta + 2a_{40}^2\beta))/(\alpha(a_{40} - \beta)),\\
b_{30} = &-(2 + 3a_{40}^2 - a_{40}\beta)/(\alpha(a_{40} - \beta)),\\
b_{21} = &-(3(a_{40} + 2a_{40}^3 + \beta))/(\alpha(a_{40} - \beta)),\\
b_{12} = &-(3a_{40}(a_{40}^3 + 2\beta + a_{40}^2\beta))/(\alpha(a_{40} - \beta)),
\end{aligned} \tag{25}$$

$$b_{03} = -(a_{40}^2(-a_{40} + 3\beta + 2a_{40}^2\beta))/(\alpha(a_{40} - \beta)),$$
$$a_{31} = a_{40}(3a_{40} + \beta),\ a_{22} = 3a_{40}^2(a_{40} + \beta),$$
$$a_{13} = a_{40}^3(a_{40} + 3\beta),\ a_{04} = a_{40}^4\beta,$$
$$b_{40} = 1,\ b_{31} = 3a_{40} + \beta,\ b_{04} = a_{40}^3\beta,$$
$$b_{22} = 3a_{40}(a_{40} + \beta),\ b_{13} = a_{40}^2(a_{40} + 3\beta).$$

Direct calculations show that in coefficients of differential system $\{(2), (25)\}$ the algebraic system $\{L_1 = 0, L_2 = 0, L_3 = 0\}$ is not compatible.

Example 4.1.

$$\begin{aligned}\dot{x} &= y + (19x(x - 2y))/(6^{1/3}95^{2/3}),\\ \dot{y} &= -(x + (6^{2/3}95^{1/3}(19x^2 - 150xy - 76y^2) - 306^{1/3}95^{2/3}x^2(x - 3y)+\\ &\quad 570x^3(x - 2y))/570).\end{aligned} \tag{26}$$

The coefficients of system (26) verify the set of conditions (25). The first two Lyapunov quantities vanish and the third one is not equal zero. This example shows that the ciclicity of the focus $(0, 0)$ in system $\{(2), (25)\}$ is at most three.

The system $\{(2), (25)\}$ has the following six exponential factors

$$\Phi_1 = \exp(x + a_{40}y),$$
$$\Phi_2 = \exp((x + a_{40}y)^2),$$
$$\Phi_3 = \exp((x + a_{40}y)^3 + 3\alpha y),$$
$$\Phi_4 = \exp((x + a_{40}y)^4 + 4y(\alpha(x + a_{40}y) + (2(1 + a_{40}^2))/(a_{40} - \beta))),$$
$$\begin{aligned}\Phi_5 = \exp(&(x + a_{40}y)^5 + 5y(\alpha(x + a_{40}y)^2 + (2(1 + a_{40}^2)(x + a_{40}y))/(a_{40} - \beta)+\\ &(4(1 + a_{40}^2)^2(1 + a_{40}\beta) + \alpha^3(a_{40} - \beta)(2 + 3a_{40}\beta - \beta^2))\\ &/(\alpha(a_{40} - \beta)^2(1 + a_{40}\beta)))),\end{aligned}$$
$$\Phi_6 = \exp((x + a40y)^6 + y(A(x + a_{40}y)^3 + B(x + a_{40}y)^2 + Cx + Dy + F)),$$

where

$$A = 6\alpha,\ B = (12(1 + a_{40}^2))/(a_{40} - \beta),$$
$$\begin{aligned}C = (&6(4\beta(1 + a_{40}^2)^2(1 + a_{40}) + \alpha^3(a_{40} - \beta)(2 + 3a_{40}\beta - \beta^2)))\\ &/(\alpha(a_{40} - \beta)^2(1 + a_{40}\beta)),\end{aligned}$$
$$\begin{aligned}D = (&3(8a_{40}(1 + a_{40}^2)^2(1 + a_{40}\beta) + (1 + a_{40}^2)\alpha^3(5 + 7a_{40}\beta - 10\beta^2)+\\ &\alpha^3(a_{40} - \beta)\beta(-13 + 3\beta^2) + \alpha^3(1 + \beta^2)(-5 + 3\beta^2)))\\ &/(\alpha(a_{40} - \beta)^2(1 + a_{40}\beta)),\end{aligned}$$
$$\begin{aligned}F = (&6(14(1 + a_{40}^2)^2\alpha^3\beta + 8(1 + a_{40}^2)^3(1 + a_{40}\beta) + 2\alpha^3\beta(1 + \beta^2)+\\ &\alpha^3(a40 - \beta)(-1 + 3\beta^2) + (1 + a_{40}^2)(\alpha^3(a_{40} - \beta)(9 - 20\beta^2)-\\ &2\alpha^3\beta(8 + 7\beta^2))))/(\alpha^2(a_{40} - \beta)^3(1 + a_{40}\beta)).\end{aligned}$$

In this way, we have proved the following two Theorems.

Theorem 4.1. *The origin* $(0,0)$ *is a center for quartic differential systems* (2) *with the line at infinity of maximal multiplicity if and only if the first three Lyapunov quantities vanish* $L_1 = L_2 = L_3 = 0$.

Theorem 4.2. *System* (2) *with the line at infinity of maximal multiplicity has a center at the origin* $(0,0)$ *if and only if its coefficients verify the set of conditions* $\{(23),\, b_{30} = 0\}$.

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(Alexandru Şubă) Moldova State University, "V. Andrunachievici" Institute of Mathematics and Computer Sciences, 5 Academiei st., Chişinău; "Ion Creangă" State Pedagogical University, 5 Gh. Iablocikin st., Chişinău, Republic of Moldova
E-mail address: `alexandru.suba@math.md`

(Olga Vacaraş) Technical University of Moldova, 9/8 Studenţilor st., Chişinău
E-mail address: `olga.vacaras@mate.utm.md`