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Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

Qualitative analysis of polynomial differential systems with the line at infinity of maximal multiplicity: exploring linear, quadratic, cubic, quartic, and quintic cases

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Abstract. This article investigates the phase portraits of polynomial differential systems with maximal multiplicity of the line at infinity. The study explores theoretical foundations, including algebraic multiplicity definitions, to establish the groundwork for qualitative analyses of dynamical systems. Spanning polynomial degrees from linear to quintic, the article systematically presents transformations and conditions to achieve maximal multiplicity of the invariant lines at infinity. Noteworthy inclusions of systematic transformations, such as Poincaré transformations, simplify analysis and enhance the accessibility of phase portraits.

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Keywords: polynomial differential system, phase portrait, infinity, multiplicity of an invariant algebraic curve, Poincaré transformation.

Studiul calitativ al sistemelor diferenţiale polinomiale cu linia de la infinit de multiplicitate maximală: studierea cazurilor liniare, pătratice, cubice, cuartice şi cuintice

Rezumat. Acest articol investighează portretele de fază ale sistemelor diferențiale polinomiale cu multiplicitatea maximă a liniei de la infinit. Studiul explorează fundamentele teoretice, inclusiv definițiile multiplicității algebrice, pentru a stabili baza pentru analize calitative ale sistemelor dinamice. Acoperind grade polinomiale de la liniar la quintic, articolul prezintă în mod sistematic transformări și condiții pentru a obține multiplicitatea maximală a dreptei invariante de la infinit. Incluziile notabile ale transformărilor sistematice, cum ar fi transformările Poincaré, simplifică analiza și îmbunătățesc accesibilitatea portretelor de fază.

Cuvinte-cheie: sistem diferenţial polinomial, portret fazic, infinit, multiplicitatea curbei algebrice invariante, transformarea Poincaré.

1. Introduction

Phase portraits serve as visual representations illustrating the temporal evolution of a differential equation system, offering insights into the long-term dynamics of the system. The complexity of the phase portrait for a polynomial differential system, characterized by the maximal multiplicity of the line at infinity, can be intricate, showcasing diverse behaviors.

The exploration of invariant algebraic curves holds significant importance in the qualitative analysis of dynamical systems [1, 2, 3, 4]. The inquiry into the maximal number of invariant straight lines within a polynomial differential system is addressed in [5]. Moreover, the incorporation of invariant straight lines in the derivation of Darboux first integrals is a notable area of investigation, as detailed in [6]. The study demonstrates that a polynomial differential system can yield a Darboux first integral when a sufficient number of invariant straight lines, considering their multiplicities, is present.

This article concentrates on the phase portraits of polynomial differential systems expressed as equations of the form:

$$\frac{dx}{dt} = P(x, y),\ \frac{dy}{dt} = Q(x, y), \tag{1}$$

where x and y are dependent variables, and t is the independent variable. The functions $P(x, y)$ and $Q(x, y)$ are polynomials in x and y. The focus is on obtaining phase portraits for polynomial differential systems with degrees $n \in \{1, 2, 3, 4, 5\}$ that possess an invariant straight line at infinity of maximal multiplicity.

Definition 1.1. *An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ is said to be an invariant algebraic curve for system* (1)*, if there exists a polynomial $K(x, y)$ such that the identity*

$$\mathbb{X}(f) = f(x, y)K(x, y)$$

holds.

To rigorously address the characterization of the invariant algebraic curve in a differential system, it becomes imperative to introduce the concept of multiplicity. Multiplicity, within this context, encompasses various facets such as algebraic, geometric, and infinitesimal type. For the specific analytical framework employed herein, we shall adopt the algebraic multiplicity as defined in reference [7].

Definition 1.2. *Let $\mathbb{C}_m[x]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x]$ of degree at most m. Then it has dimension $R = \binom{n+m}{n}$. Let $v_1, v_2, \ldots, v_R$ be a base of $\mathbb{C}_m[x]$. If k is the greatest positive integer such that the k-th power of f divides $\det M_R$, where*

$$M_R = \begin{pmatrix} v_1 & v_2 & \ldots & v_R \\ \mathbb{X}(v_1) & \mathbb{X}(v_2) & \ldots & \mathbb{X}(v_R) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{X}^{R-1}(v_1) & \mathbb{X}^{R-1}(v_2) & \ldots & \mathbb{X}^{R-1}(v_R) \end{pmatrix},$$

then the invariant algebraic curve f of degree m of the vector field $\mathbb{X}$ has algebraic multiplicity k.

Various methodologies exist for examining the behaviour at infinity within a polynomial differential system. In this context, I will employ a straightforward approach, leveraging one of the Poincaré transformations. Through this transformation, the infinity locus is effectively mapped onto one of the axes in the newly defined coordinates, thereby assuming the role of an invariant straight line within the finite part of the phase plane. Subsequently, the analytical tools elucidated earlier will be applied.

2. Linear and quadratic differential systems

A general linear differential system has the form:

$$\begin{cases} \dot{x} = a_{00} + a_{10}x + a_{01}y, \\ \dot{y} = b_{00} + b_{10}x + b_{01}y, \end{cases} \tag{2}$$

Utilizing the Poincaré transformation, denoted as $x \to \frac{1}{z}, y \to \frac{y}{z}$, followed by a subsequent adjustment for enhanced visual clarity $y \to x$ and $z \to y$, the transformed system is expressed as follows:

$$\begin{cases} \dot{x} = -b_{10} + (a_{10} - b_{01})\,x - b_{00}y + a_{01}x^2 + a_{00}xy, \\ \dot{y} = y\,(b_{00} + b_{10}x + b_{01}y)\,. \end{cases} \tag{3}$$

In this representation, the variable y corresponds to the invariant straight line characterizing the infinity of the system specified by equation (2).

Our objective is to achieve the maximum multiplicity for the system (2), which is equivalent to ensuring that the system (3) also attains its maximum multiplicity. Notably, the degree of the polynomial $\det M_r$ is equal to 3

$$\det M_R = A_1(x)y + A_2(x)y^2 + A_3(x)y^3,$$

implying that the system (2) can theoretically exhibit a multiplicity of up to 3, where

$$A_1\,(x) = b_{10}\,(a_{10}b_{01} - a_{01}b_{10}) + (b_{01} - a_{10})\,(a_{10}b_{01} - a_{01}b_{10})\,x + a_{01}\,(a_{01}b_{10} - a_{10}b_{01})\,x^2,$$

$$A_2\,(x) = -a_{10}^2b_{00} + a_{10}b_{00}b_{01} + a_{00}a_{10}b_{10} - 2a_{01}b_{00}b_{10} + a_{00}b_{01}b_{10} + (-a_{01}a_{10}b_{00} -$$
$$-a_{00}a_{10}b_{01} - a_{01}b_{00}b_{01} + a_{00}b_{01}^2 + 2a_{00}a_{01}b_{10}\Big)\,x,$$

$$A_3\,(x) = -a_{00}a_{10}b_{00} - a_{01}b_{00}^2 + a_{00}b_{00}b_{01} + a_{00}^2b_{10}.$$

Requiring both $A_1(x)$ and $A_2(x)$ to be zero, while simultaneously ensuring $A_3(x) \neq 0$, entails solving a straightforward system of algebraic equations. Upon resolution, it is shown that the infinity of the system (2) possesses a multiplicity of 3. The system can be expressed in the following form:

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x. \end{cases} \tag{4}$$

The phase portrait on the Poincaré disk is presented in Figure 1.a) for the given system. Employing a parallel methodology, we confirm that the infinity of the general quadratic differential system (5)

$$\begin{cases} \dot{x} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{cases} \tag{5}$$

achieves a multiplicity of 5 and can be expressed as

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x^2. \end{cases} \tag{6}$$

The corresponding phase portrait is depicted in Figure 1.b).

As stated in [8], the maximal multiplicity of the line at infinity for cubic systems is identified as seven, and these systems can be reformulated into the following two configurations:

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x^3 + ax, \end{cases} \qquad a \in \mathbb{R}; \tag{7}$$

and

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = x^3 + 2y. \end{cases} \tag{8}$$

The phase portraits depicted in Figure 1.c) and 1.d) for these two systems were demonstrated in [9].

3. Quartic and quintic differential systems

As outlined in [10], a quartic polynomial differential system with maximal multiplicity can be transformed into the following canonical form:

$$\begin{cases} \dot{x} = -3x + ay^4, \\ \dot{y} = y, \quad a > 0. \end{cases} \tag{9}$$

This system features an invariant line at infinity with a multiplicity of 10. Referring to [11], the phase portrait of this system can be constructed. However, to facilitate this analysis, it is necessary to relocate the singular points at infinity to the ends of the Oy axis by implementing the transformation $x \to y$, $y \to x$ (Figure 1.e)).

As indicated in [12], a quintic polynomial differential system with the line at infinity exhibiting maximal multiplicity can be written into the following form:

$$\begin{cases} \dot{x} = x, \quad a \neq 0, \\ \dot{y} = -4y + ax^5. \end{cases} \tag{10}$$

The system's structure remains unaltered under the transformations $x \to x$, $y \to -y$, $a \to -a$, with the additional condition $a > 0$ imposed for generality. To align its phase portrait with others, we apply the transformation $x \to y$, $y \to x$ to system (10), resulting in the following transformed system:

$$\begin{cases} \dot{x} = -4x + ay^5, \\ \dot{y} = y, \quad a > 0. \end{cases} \tag{11}$$

The phase portrait depicted in Figure 1.f) is obtained from the analysis presented in [9].

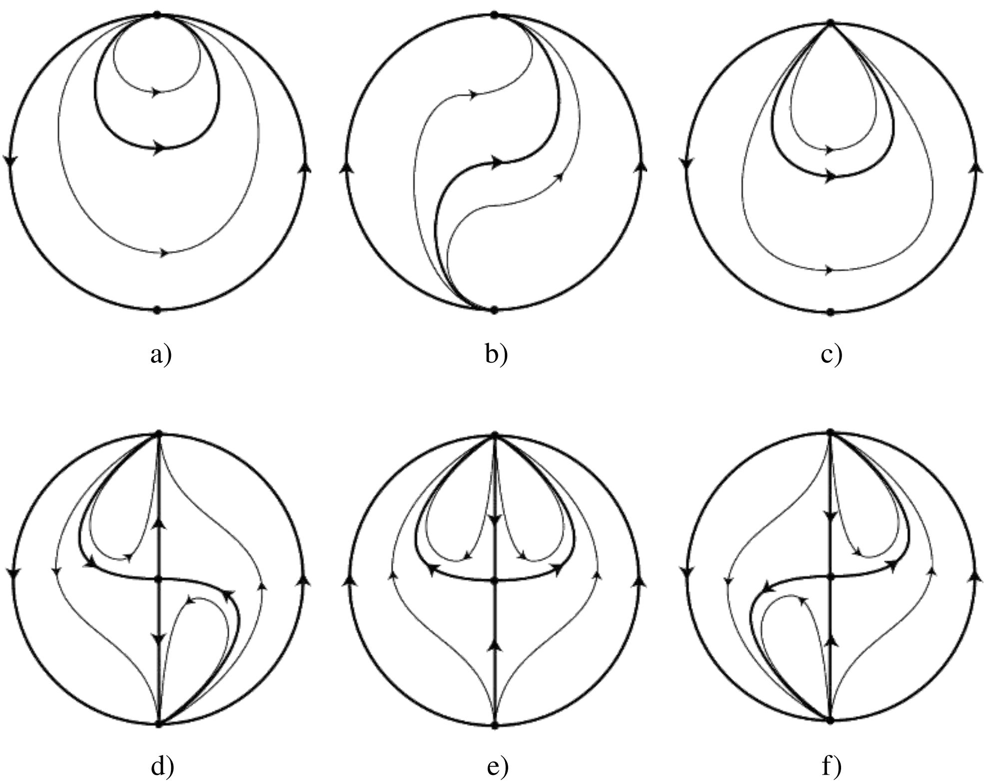

Figure 1. Phase portraits of all polynomial ($n \leq 5$) differential systems with infinity of maximal multiplicity.

4. Conclusions

This article explores phase portraits for polynomial differential systems, emphasizing the significance of invariant algebraic curves, particularly those associated with maximal multiplicity at the line at infinity. Theoretical foundations, including algebraic multiplicity definitions, lay the groundwork for qualitative dynamical system analysis.

Examining polynomial degrees from linear to quintic, the article systematically presents transformations and conditions to achieve desired invariant structures at infinity, offering nuanced insights into the dynamics of polynomial differential equations. Systematic transformations, like Poincaré transformations, simplify analysis and enhance accessibility to phase portraits.

Encompassing various polynomial degrees, including linear, quadratic, cubic, quartic, and quintic systems, the article contributes to a comprehensive understanding of the interplay between invariant algebraic curves, multiplicity, and resulting phase portraits in polynomial differential dynamics.

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