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RELIABILITY OF SERIAL-PARALLEL NETWORKS VS RELIABILITY OF PARALLEL-SERIAL NETWORKS WITH CONSTANT NUMBERS OF SUB-NETWORKS AND UNITS

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Abstract. In this paper it is performed, based on dynamic models, a comparative analysis of the reliability of two types of networks: serial-parallel and parallel-serial networks when the numbers of subnets and units in each subnet are predefined, constant numbers, but also when the lifetimes of the units are independent random variables. The equations for calculating the reliability of the related networks have been deduced. These functions are deduced for the dynamic model which is less studied and it is prooved to be relevant to the static model too. All equations are demonstrated and graphically illustrated in some examples. A few examples are analyzed graphically for different values of the number of units in the subnet and the number of subnets. This paper contains four different network topology models which are also analyzed by equations and graphically. The mathematical model described and the deduced equations will serve as a basis for the subsequent analysis of the dynamic networks of various topologies and various types of random variables that describe the lifetimes of the units of the analyzed system.

Keywords: *cumulative distribution function, distributions, global maximum, lifetime, survival functions.*

Rezumat. Pe baza modelelor dinamice în lucrare se realizează o analiză comparativă a fiabilității a două tipuri de rețele: rețele serial-paralele și paralel-seriale, când sunt predefinite numerele de subrețele și unități din fiecare subrețea, numere constante, dar și când duratele de viață ale unităților sunt variabile aleatorii independente. Au fost deduse ecuațiile pentru calcularea fiabilității rețelelor aferente. Aceste funcții sunt deduse pentru modelul dinamic, care este mai puțin studiat și care a fost demonstrat anterior drept relevant pentru modelul static. Toate ecuațiile sunt demonstrate și ilustrate grafic în câteva exemple. Unele exemple sunt analizate grafic pentru diferite valori ale numărului de unități din subrețea și ale numărului de subrețele. Această lucrare conține patru modele diferite de topologie de rețea care sunt, de asemenea, analizate prin ecuații și grafic. Modelul matematic descris și ecuațiile deduse vor servi ca bază pentru analiza ulterioară a rețelelor dinamice de

diverse topologii și diferite tipuri de variabile aleatorii, care descriu durata de viață ale unităților sistemului analizat.

Cuvinte cheie: *funcție de distribuție cumulativă, distribuții, maxim global, durată de viață, funcții de supraviețuire.*

1. Introduction

First of all, let us observes that many mathematical models in Network's Reliability deal with serial and parallel Networks as a subsystem of Network with more complex structure/topology considered in many works as [1-5]. At the same time, it can happen that due to the different topologies / structures of two networks made up of some and the same elements, they have different reliability [6,7]. If for elementary networks, such as serial and parallel networks, it is obvious that the last are, under the stated conditions, more reliable, but in the case of more complex structures, the answer is not so obvious. Let us take, for beginning, two standard network types: serial-parallel named scheme (*A*) and parallel-serial named scheme (*B*), according to the figure 1.

A - Serial-Parallel Network scheme ***B*** - Parallel-Serial Network scheme

Figure 1. Serial-parallel and parallel-serial schemes.

Reliability of serial-parallel networks vs parallel networks it was well studied in the Chapter 3 of the book [8], but their conclusion was founded on the base of the static models. Next, we will adopt dynamic mathematical models for the networks represented above. More exactly, we suppose the following: the lifetime of each network's unit is described as a nonnegative r.v. whose cumulative distribution function (c.d.f.) is known and lifetimes of such units are independent random variables (i.r.v.). Total number of units N_k in the k-th sub-network is a constant (natural) number, where k = $1, 2, ..., M$ and number M of all sub-networks is also a constant (natural) number. At the same time, each unit becomes irreplaceable at the time of its fall. On the other hand, reliability of the network coincides with the survival function, i.e., with the probability that the network will survive a longer time than x, which coincides, in fact, with the tail of c.d.f. for the lifetime of the entire network which we find in other specialist books [9-15].

In the following we are interested in:

- obtaining equations for calculation of the reliability of networks of type *A* and *B* under different conditions;

- to perform, on the basis of this equations, a comparative analysis of the networks from the point of view of their reliability in some particular cases.

2. Reliability of serial-parallel and parallel-serial networks where numbers of sub-networks and units in each sub-network are integers.

So, as a starting point in the study of reliability of more complicated mathematical models of networks of type ***A*** or ***B*** let us take the case when the numbers N_k, k = 1, 2, ..., M and M are integers. We consider that the lifetimes of all units compounding the Network are non-negative, independent random variables (i.r.v.) X_{kj} with c.d.f. $F_{kj}(x)$ for *j - th* unit of the *k - th* sub-network, $j = \overline{1, N_k}$, $k = \overline{1, M}$. Then we observe that, for each type of Network, the lifetime of sub-networks is i.i.d.r.v $Y_k = min\ (X_{k1}, X_{k2}, \ldots, X_{kN_k})$, in the case of parallel-serial Networks, or $Y_k = max\ (X_{k1}, X_{k2}, \ldots, X_{kN_k})$, in the case of serial-parallel Networks, $k = \overline{1, M}$. So, the lifetime of entire Network is the r.v. $U = min\ (Y_1, Y_2, \ldots, Y_M)$, in the case of serial-parallel Networks or is the r.v.

$V = max\ (Y_1, Y_2, \ldots, Y_M)$, in the case of parallel-serial Networks. But c.d.f of the r.v. $min\ (X_{k1}, X_{k2}, \ldots, X_{kN_k})$ coincides with $1 - \prod_{j=1}^{N_k}(1 - F_{kj}(x))$ and c.d.f. of the r.v. $max\ (X_{k1}, X_{k2}, \ldots, X_{kN_k})$ coincides with $\prod_{j=1}^{N_k} F_{kj}(x)$, *k*=1, 2, ..., *M*. In the above described conditions, we have that c.d.f.:

$$F_U(x) = P(U \le x) = 1 - \prod_{k=1}^{M}\left[1 - \prod_{j=1}^{N_k} F_{kj}(x)\right] \quad (1)$$

$$F_V(x) = P(V \le x) = \prod_{k=1}^{M}\left[1 - \prod_{j=1}^{N_k}(1 - F_{kj}(x))\right] \quad (2)$$

On the base of this equations we may: a) calculate survival / reliability function of serial-parallel and parallel-serial networks with integer number of subnetworks and integer number of units in each subnetwork and b) formulate the algorithm by which we can determine which Network is more reliable, ***A*** or ***B***? Indeed:

a) Using equations (l) - (2) and description of survival / reliability function, we find that survival / reliability functions of serial-parallel and parallel-serial networks may be calculate, respectively, by the following equations

$$S_{S-P}(X) = P(U > x) = 1 - F_U(x) = \prod_{k=1}^{M}\left[1 - \prod_{j=1}^{N_k} F_{kj}(x)\right] \quad (3)$$

$$S_{P-S}(X) = P(V > x) = 1 - F_V(x) = 1 - \prod_{k=1}^{M}\left[1 - \prod_{j=1}^{N_k}(1 - F_{kj}(x))\right] \quad (4)$$

b) Formally, to determine which kind of the Network is more reliable, we must compare survival / reliability functions $S_{U_M}(x)$, $S_{V_M}(x)$ for concrete lifetime distributions $F_{kj}(x)$, $j = \overline{1, N_k}$, $k = \overline{1, M}$.

As example, let us take the particular case when lifetimes of all units are independent, identically distributed random variables (i.i.d.r.v.) X_{kj}, i.e., c.d.f. $F_{kj}(x) = F(x)$, $j = \overline{1, N_k}$, $k = \overline{1, M}$. Then, from Eq. (3) and Eq. (4), we deduce that the corresponding survival/reliability functions of serial-parallel *A* and parallel-serial *B* networks are

$$S_{S-P}(x) = P(U > x) = 1 - F_U(x) = \prod_{k=1}^{M}[1 - (F(x))^{N_k}] \quad (5)$$

$$S_{P-S}(x) = P(V > x) = 1 - F_V(x) = 1 - \prod_{k=1}^{M}[1 - (1 - F(x))^{N_k}] \quad (6)$$

In fact, in this particular case, Eq. (5) and Eq. (6) show us that the property of one network to be more reliable than the other does not depend of the lifetime distribution function. This is a consequence of the following

Proposition 1. *If lifetimes of all units are i.i.d.r.v., then survival / reliability functions* $S_{S-P}(x) \ge S_{P-S}(x)$ *or* $S_{S-P}(x) \le S_{P-S}(x)$ *if and only if, respectively functions* $g_{S-P}(M, N_1, N_2, \ldots, N_M; q) \ge g_{P-S}(M, N_1, N_2, \ldots, N_M; q)$ *or* $g_{S-P}(M, N_1, N_2, \ldots, N_M; q) \le g_{P-S}(M, N_1, N_2, \ldots, N_M; q)$ *for every* $q \in [0, 1]$,
where

$$g_{S-P}(M, N_1, N_2, \ldots, N_M; q) = \prod_{k=1}^{M}[1 - q^{N_k}] \text{ and}$$

$$g_{P-S}(M, N_1, N_2, \dots, N_M; q) = 1 - \prod_{k=1}^{M}[1 - (1-q)^{N_k}].$$

Proof. Before comparing the c.d.f. $F_U(x)$ and $F_V(x)$, let us take into account their characteristic properties. So, we observe that both functions are monotonous non-decreasing functions and $0 \le F_U(x) \le 1, 0 \le F_V(x) \le 1$, for every $x \in (-\infty, +\infty)$. If we denote by $q = F(x)$ for fixed x, then, according to the Eq. (5) and Eq. (6), the problem of the comparison of this c.d.f. will be equivalent to comparison of the functions

$$\prod_{k=1}^{M}[1 - q^{N_k}] \text{ and } 1 - \prod_{k=1}^{M}[1 - (1-q)^{N_k}], \textit{for every } q \in [0,1].$$

So, the problem of comparison of the survival / reliability functions $S_{S-P}(x), S_{P-S}(x)$ for every $x \in (-\infty, +\infty)$ will be equivalent to the comparison of the functions $g_{S-P}(M, N_1, N_2, \dots, N_M; q)$, $g_{P-S}(M, N_1, N_2, \dots, N_M; q)$.

Remark 1. According to the proof, we observe, in fact, that the functions $g_{S-P}(M, N_1, N_2, \dots, N_M; q)$, $g_{P-S}(M, N_1, N_2, \dots, N_M; q)$ represents, respectively, survival / reliability functions of above mentioned Networks when lifetime distribution *F(x)* coincides with Uniform distribution on the interval [0, 1].

Before we give some examples, we mention that in case $M = 1$, i.e., network A becomes a serial type network and network B becomes a parallel type, the result is obvious. Specifically, parallel networks are always more reliable than serial networks. Therefore, in the following examples we will consider that the number M of subnets is greater than one.

Example 1. *a)* Let us consider serial-parallel and parallel-serial networks with number of subnetworks $M = 3$ and numbers $N_1=4$, $N_2=2$, $N_3=2$ of units in corresponding subnetworks. Then the following graphical representation of

$$g_{S-P}(3,4,2,2;q) = (1-q^4)(1-q^2)^2 \ \ and$$

$$g_{P-S}(3,4,2,2;q) = 1 - (1-(1-q)^4)(1-(1-q)^2)^2$$

show us that $g_{S-P}(3,4,2,2;q) \ge g_{P-S}(3,4,2,2;q)$ ***not*** for every $q \in [0,1]$, reprezented in figure 2.

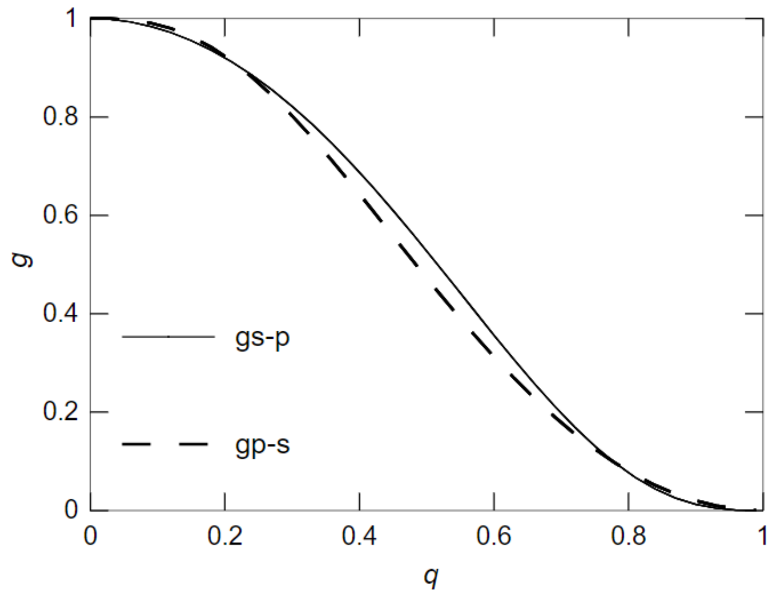

Figure 2. Grafics for g_{S-P} and g_{P-S} presented in the example 1*a*.

This means that the reliability of both networks cannot be unequivocally compared. More than that

b) Considering serial-parallel and parallel-serial networks with number of subnetworks $M=3$ and numbers $N_1=2$, $N_2=2$, $N_3=2$ of units in corresponding subnetworks, we show graphically in figure 3 that

$$g_{S-P}(3,2,2,2;q) = (1-q^2)^3 \ \le \ g_{P-S}(3,2,2,2;q) = [1-(1-(1-q)^2)^3]$$

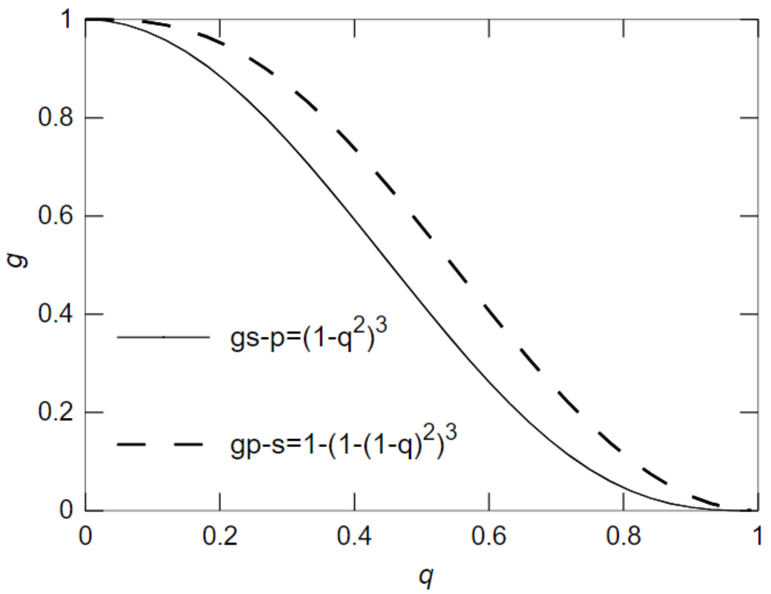

Figure 3. Grafics for g_{S-P} and g_{P-S} presented in the example $1b$.

So, in this case, parallel-serial Network is more reliable than serial-parallel Network.

Remark 2. According to the Proposition 1 and Remark 1 we note that the conclusions regarding the reliability of the type ***A*** network vs the reliability of the type ***B*** network in dynamic case remain valid in static case, i.e., when all units have one and the same probability of fall equal to q, where 0 <q <1. Moreover, both examples *a)* and *b)* above refer to the case where $min(N_1, N_2, \dots, N_M) < M$, for which shows that the reliability of the Network ***A*** vs Network ***B*** it cannot be uniquely characterized.

3. Results

Now, we can prove the following

Proposition 2. *For M > 1, if the lifetimes of the network units are i.i.d.r.v. and* $min(N_1, N_2, \dots, N_M) \ge M$, *then the survival / reliability functions* $S_{S-P}(x) \ge S_{P-S}(x)$, *i.e., networks of type A are more reliable than networks of type B, regardless of the lifetime distribution of their units.*

Proof. Indeed, according to the Proposition 1, it is sufficient to prove that

$$\prod_{k=1}^{M}[1-q^{N_k}] \ \ge \ 1-\prod_{k=1}^{M}[1-(1-q)^{N_k}],$$

i.e., that

$$g(M, N_1, N_2, \dots, N_M; q) = \prod_{k=1}^{M}[1-q^{N_k}] + \prod_{k=1}^{M}[1-(1-q)^{N_k}] \ge 1$$

$for\ every\ q \in [0,1]$, as soon as $min(N_1, N_2, \dots, N_M) \geq M$. The fact that $q \in [0,1]$, implies inequality

$$g(M, N_1, N_2, \dots, N_M; q) \geq h(M; q) = [1 - q^M]^M + [1 - (1 - q)^M]^M.$$

The following graphical representation of *h(M; q)*, as a function of *q* for $M \in \{2, 3, 4\}$ represented in figure 4 , suggest as that *h*(*M; q*) ≥ 1 with the local minimal value 1 in the points *q* = 0 and *q* = 1.

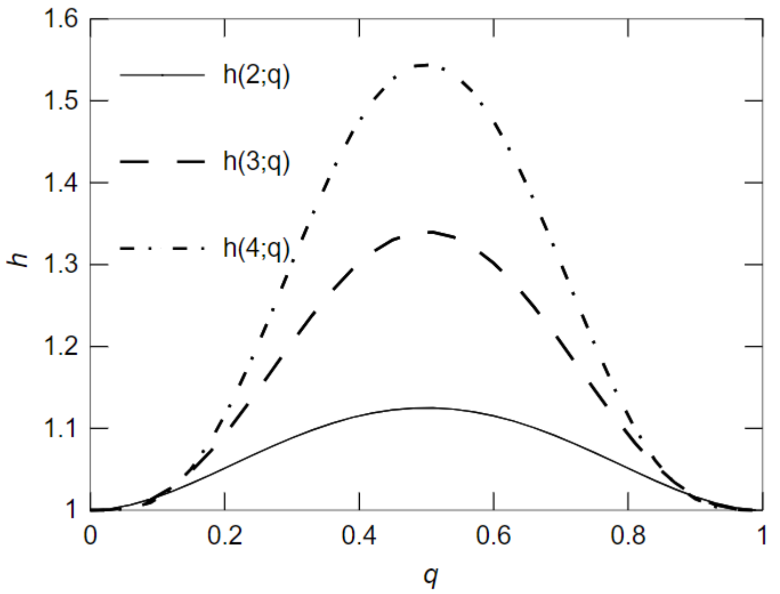

Figure 4. Grafics for *h(M; q)* presented in the proposition 2.

Let us study the properties of *h(M; q)*, as a function of q for fixed *M* > 1.
His derivative with respect to variable q is equal to

$$\frac{dh(M;q)}{dq} = M^2[(1 - (1 - q)^M)^{M-1}(1 - q)^{M-1} - (1 - q^M)^{M-1}q^{M-1}]$$

and equation $\frac{dh(M;q)}{dq} = 0$ have a single solution q = 1/2 in the interval (0, 1).

Furthermore, $\frac{dh(M;q)}{dq} > 0\ for\ 0 < q < \frac{1}{2}, and\ \frac{dh(M;q)}{dq} < 0\ for\ \frac{1}{2} < q < 1$. This means that for each integer *M* > 1 the point *q* = 1/2 is a point of global maximum of *h(M; q)* and this function is a concave of on the interval [0, 1] (see the previous graphical representation). So, *h(M; q)* ≥ 1, because *h*(*M*; 0) = *h*(*M*; 1) = 1 are the points of minimum of this concave with the global maximum in the point *q* = 1/2. This implies that $g(M, N_1, N_2, \dots, N_M; q) \geq h(M; q) \geq 1$, which means in this case that serial-parallel network is more reliable than parallel- serial network.

Example 2. a) Let us consider serial-parallel and parallel-serial networks with number of subnetworks *M* = 2 and numbers $N_1 = 2$, $N_2 = 2$ of units in corresponding subnetworks. Then the following graphical representation in figure 5.

$$g_{S-P}(2,2,2;q) = (1 - q^2)^2\ and\ g_{P-S}(2,2,2;q) = 1 - (1 - (1 - q)^2)^2,$$

confirm that $g_{S-P}(2,2,2;q) \geq g_{P-S}(2,2,2;q)$ for every $q \in [0,1]$.

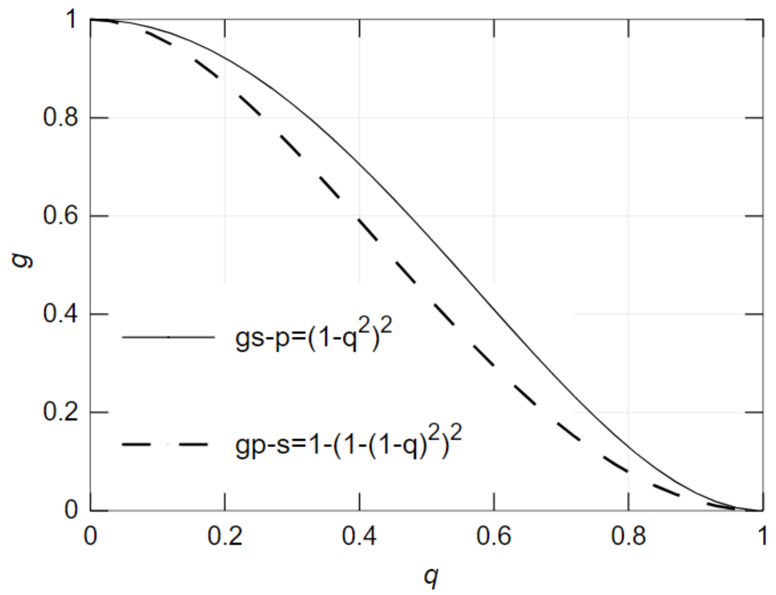

Figure 5. Grafics for g_{S-P} and g_{P-S} presented in the example 2a.

b) Now, let us consider serial-parallel and parallel-serial networks with number of subnetworks M = 2 and numbers N_1 = 5, N_2 = 8 of units in corresponding subnetworks. Then the following graphical representation of

$g_{S-P}(2,5,8;q) = (1-q^5)(1-q^8)$ and $g_{P-S}(2,5,8;q) = 1-(1-(1-q)^5)(1-(1-q)^8)$,

confirm too that $g_{S-P}(2,5,8;q) \geq g_{P-S}(2,5,8;q)$ for every $q \in [0,1]$ reprezented in figure 6.

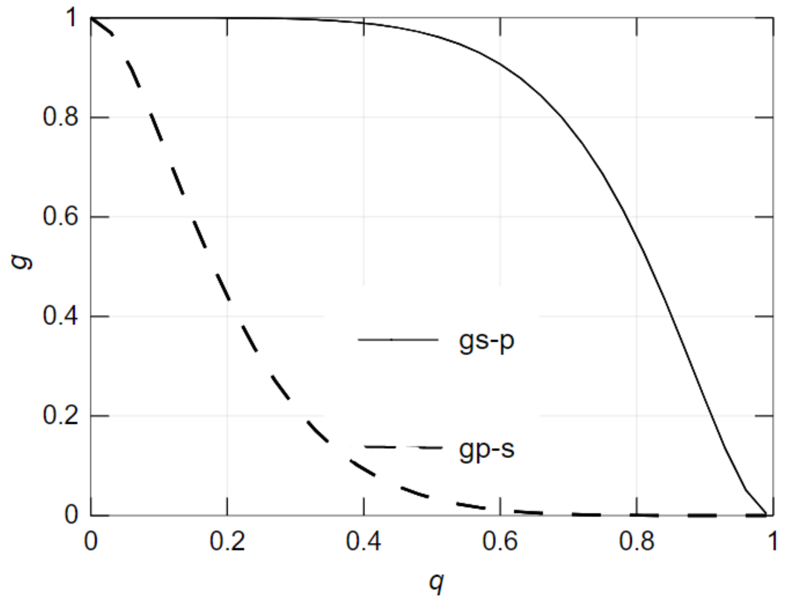

Figure 6. Grafics for g_{S-P} and g_{P-S} presented in the example 2b.

4. Discussion: comparative analysis with other models

The calculation Eq. (5) and Eq. (6) for the reliability of type ***A*** and ***B*** networks allow us to solve, in a dynamic case, the problem of reliability of the models proposed and researched in the paper [8], but in a static case. These are the following models:

***C*. Serial-parallel.** The number of subnets is equal to N> 1 and the number of units in each subnet is equal to M> 1, the subnets being connected in serial and the units in parallel;

D*. Parallel-serial.** The number of subnets is equal to M> 1 and the number of units in each subnet is equal to N> 1, the subnets being connected in parallel and the units in serial. In both models ***C and ***D*** it is assumed that the lifetimes of the units are i.i.d.r.v. with c.d.f. *F(x)*. Then, from Eq. (5) and Eq. (6), we deduce that the corresponding survival/reliability functions of serial-parallel ***C*** and parallel-serial ***D*** networks are

$$S_{S-P}(x) = [1-(F(x))^M]^N \tag{7}$$

$$S_{P-S}(x) = 1-\left[1-\left(1-F(x)\right)^N\right]^M \tag{8}$$

Using the same techniques of mathematical analysis of the survival/reliability function (7)-(8) we can prove

Proposition 3. *For every M > 1 and N>1, if the lifetimes of the network units are i.i.d.r.v., then the survival / reliability functions* $S_{S-P}(x) \geq S_{P-S}(x)$, *i.e., networks of type **C** are more reliable than networks of type **D**, regardless of the lifetime distribution of their units.*

In fact, this sentence extends the authors' conclusions [8] to the ***C*** and ***D*** dynamic models. In particular, example 2 a) above confirms these conclusions. The example below illustrates that these conclusions are also valid in M> N or M <N cases.

Example 3. a) Let us consider serial-parallel and parallel-serial networks ***C*** and ***D*** when *N* = 3 and *M=2*. Then the following graphical representation of

$$g_{S-P}(3,2,2,2;q) = (1-q^2)^3 \; and \; g_{P-S}(2,3,3\,;q) = 1-(1-(1-q)^3)^2,$$

confirm that $g_{S-P}(3,2,2,2;q) \geq g_{P-S}(2,3,3;q)$ for every $q \in [0,1]$.

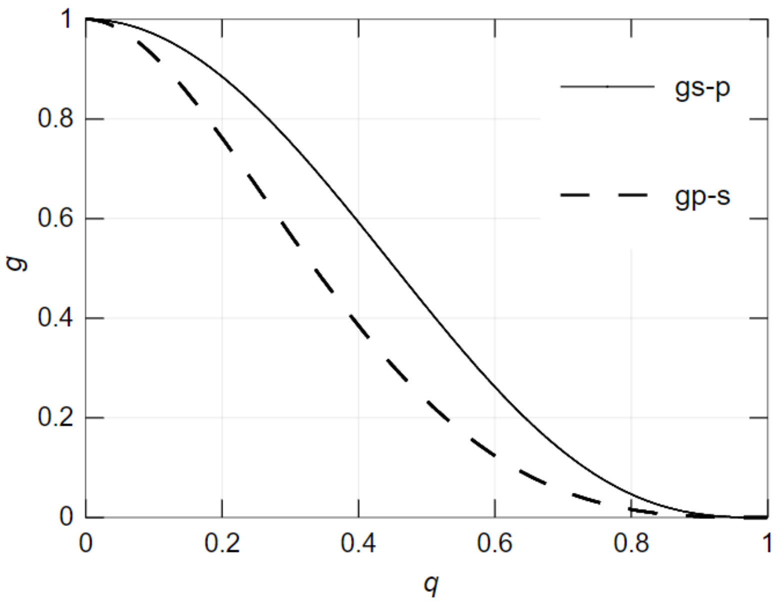

Figure 7. Grafics for g_{S-P} and g_{P-S} presented in the example 3a.

b) Conversely, let us consider serial-parallel and parallel-serial networks ***C*** and ***D*** when *N* = 2 and *M=3*. Then the following graphical representation of

$$g_{S-P}(2,3,3;q) = (1-q^3)^2 \; and \; g_{P-S}(3,2,2,2\,;q) = 1-(1-(1-q)^2)^3,$$

confirm that $g_{S-P}(2,3,3;q) \geq g_{P-S}(3,2,2,2\,;q)$ for every $q \in [0,1]$.

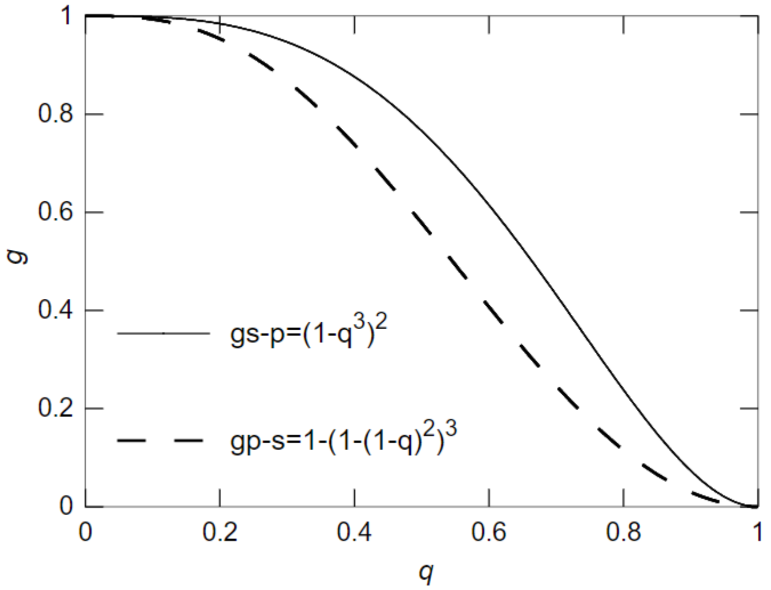

Figure 8. Grafics for g_{S-P} and g_{P-S} presented in the example 3b.

5. Conclusions

Equations for calculating their reliability was deduced for the serial-parallel and parallel-serial networks when numbers of the units and sub-networks are constant numbers, considering that the lifetimes of the units are independent random variables.

In the particular case, when the lifetimes of the units are i.i.d.r.v. we have proved (see Proposition 1), that the property of one network to be more reliable than the other does not depend of the lifetime distribution function.

The examples provided and Proposition 2 show us that, in this particular case, the serial-parallel networks of type ***A*** are always more reliable than the parallel-serial networks of type ***B*** as soon as the number *M* of the subnetworks is greater than 1 and the subnetwork with the smallest number of units exceeds this number *M*.

Our deductions based on dynamic models generalize, in fact, through Proposition 3, the conclusions of the other authors conclusions based on static models.

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