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GENERALIZED HIGHER DERIVATIONS

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Abstract

A type of generalized higher derivation consisting of a collection of self-mappings of a ring associated with a monoid, and here called a *D-structure*, is studied. Such structures were previously used to define various kinds of 'skew' or 'twisted' monoid rings. We show how certain gradings by monoids define D-structures. The monoid ring defined by such a structure corresponding to a group-grading is the variant of the group ring introduced by Năstăsescu, while in the case of a cyclic group of order two, the form of the D-structure itself yields some gradability criteria of Bakhturin and Parmenter. A partial description is obtained of the D-structures associated with infinite cyclic monoids.

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1. Introduction

All rings considered are associative with identity, and ring homomorphisms preserve identities.

A *higher derivation of rank m* (respectively, *of infinite rank*) on a ring R is a sequence $d_0, d_1, \ldots, d_m$ (respectively, $d_0, d_1, d_2, \ldots$) of additive endomorphisms of R such that $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$ for each relevant n and for all $a, b \in R$. The defining condition ensures that in both cases d_0 is an endomorphism and d_1 is a (d_0, d_0)-derivation (whence a derivation when d_0 is the identity map). This concept is quite well established and seems to have been introduced (for fields) by Hasse and Schmidt [4]. Higher derivations are closely related to homomorphisms from R to $R[X]/(X^m)$ (in the rank-m case) and to $R[[X]]$ (in the infinite-rank case) and (at least when d_0 is the identity) the d_i can be described in terms of derivations [1, 5, 6].

In [3] the first author introduced what we claim is a generalization of higher derivations. We first recall the definition and then give a justification for the claim.

Let G be a monoid with identity e, R a ring with identity 1, and for each $x, y \in G$ let $\sigma_{x,y} : R \to R$ be a function. We require the set of $\sigma_{x,y}$ to satisfy the following condition, where x, y, z are arbitrary elements of G and a, b of R.

Condition A.

(0) For each $x \in G$ and $a \in R$, we have $\sigma_{x,y}(a) = 0$ for almost all $y \in G$.
(i) Each $\sigma_{x,y}$ is an additive endomorphism.
(ii) $\sigma_{x,y}(ab) = \sum_{z\in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$.
(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$.
(iv_1) $\sigma_{x,y}(1) = 0$ if $x \neq y$.
(iv_2) $\sigma_{x,x}(1) = 1$.
(iv_3) $\sigma_{e,x}(a) = 0$ if $x \neq e$.
(iv_4) $\sigma_{e,e}(a) = a$.

Definition 1.1. A collection of functions $\sigma_{x,y}$ satisfying Condition A is called a *D*-structure.

Condition A(iii) obviously has at least a superficial similarity to the defining condition for a higher derivation, but a closer examination reveals a more significant connection. Let $d_0, d_1, d_2, \ldots$ be a higher derivation of infinite rank in which for convenience *we take d_0 to be the identity function*. We make use of the *additive* monoid $\mathbb{N}$. For $x, y \in \mathbb{N}$ let $\sigma_{x,y} = (x!/y!)d_{x-y}$ for $x \geq y$ and the zero function for $x < y$. We show that the $\sigma_{x,y}$ satisfy most of Condition A.

Clearly (i) is satisfied. If $x \geq y \in \mathbb{Z}^+$ and $a, b \in R$, then

$$\begin{aligned}\sigma_{x,y}(ab) &= \frac{x!}{y!}d_{x-y}(ab) = \frac{x!}{y!}\sum_{t=0}^{x-y} d_{x-y-t}(a)d_t(b) = \frac{x!}{y!}\sum_{y\leq z\leq x} d_{x-z}(a)d_{z-y}(b)\\ &= \sum_{y\leq z\leq x} \frac{x!}{z!}d_{x-z}(a)\frac{z!}{y!}d_{z-y}(b) = \sum_{y\leq z\leq x} \sigma_{x,z}(a)\sigma_{z,y}(b),\end{aligned}$$

while for $x < y$ everything is zero, so we have (ii). It is easy to prove by induction that $d_n(1) = 0$ for all $n \geq 1$ so (recalling that d_0 is the identity) for $x > y$ we have $\sigma_{x,y}(1) = (x!/y!)d_{x-y}(1) = 0$ while $\sigma_{x,x}(1) = (x!/x!)d_0(1) = 1$, whence ($\mathrm{iv}_1$) and ($\mathrm{iv}_2$) hold. Observing that $\sigma_{0,x}$ is zero for all $x > 0$ and $\sigma_{0,0} = d_0$, we see that the rest of (iv) is valid too.

What about (iii)? A higher derivation $d_0, d_1, d_2, \ldots$ is *iterative* [4] if $d_i \circ d_j = \binom{i+j}{i}d_{i+j}$ for each i, j. If (iii) is satisfied then, in particular,

$$(i + j)!d_{i+j} = \frac{(i + j)!}{0!}d_{i+j} = \sigma_{i+j,0} = \sigma_{i,0} \circ \sigma_{j,0} = i!d_i \circ j!d_j,$$

so

$$d_i \circ d_j = \frac{(i + j)!}{i!\,j!} = \binom{i + j}{i}d_{i+j}$$

and thus the higher derivation is iterative. Conversely, if the higher derivation is iterative, then for each x, y, z we have

$$\begin{aligned}\sum_{u+v=z} \sigma_{x,u} \circ \sigma_{y,v} &= \sum_{u+v=z} \frac{x!}{u!}d_{x-u}\frac{y!}{v!}d_{y-v}\\ &= \sum_{u+v=z} \frac{x!}{u!}\frac{y!}{v!}\binom{x - u + y - v}{x - u}d_{x-u+y-v}\end{aligned}$$