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Computer Simulation of Multi-optional Decisions

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Abstract

The formula for the estimation of average disproportion of seats allocation using Hamilton method is obtained. It is shown that, by proportionality of voters' will representation in the final multi-optional decision, the best from the d'Hondt, Huntington-Hill and Sainte-Laguë methods, is the last one. Also, the use of Sainte-Laguë method is easier than that of complemented Webster method. Moreover, the proposed monotone adapted Sainte-Laguë method is considerably better than the Huntington-Hill one. So, for apportionment in the United States Congress House of Representatives, the adapted Sainte-Laguë method is more convenient than the used from 1941 year Huntington-Hill method.

Keywords: disproportion, votes-decision methods, computer simulation, comparison, predicting disproportionality

1 Introduction

The main issue of multi-optional decision-making systems with proportional representation (PR) is the disproportion of voters' will representation in final decision. As criteria of disproportionality it is opportune to use the Average relative deviation index (ARD) I_d [2]. Its minimum value is ensured by Hamilton (Hare) method [1, 4]. However, this method is not immune to the Alabama, of Population and of New State paradoxes [1]. Therefore, in many cases they deny its application in the benefit of monotonous divisor methods, such as d'Hondt, Sainte-Laguë and Huntington-Hill ones [1, 4]. At the same time, it is not strictly determined which of these "votes-decisions" (VD) methods is more convenient.

Qualitative comparison of Hamilton, Huntington-Hill, d'Hondt, Sainte-Laguë and Mixed VD methods by disproportionality (I_d), quota rule, immunity to paradoxes and non-favoring parties, basing on results from [1, 3-5], are systemized in [7]. Quantitative comparisons of these five methods for particular cases, by average disproportion and non-favoring parties, were done in [1, 3, etc.]. Some results of comparison of Hamilton, d'Hondt and Mixed VD methods by computer simulation are described in [7].

Known results of comparing VD methods are extended, in this paper, by computer simulation, using the elaborated application SIMOD. The average value of I_d index for optimal solutions using Huntington-Hill and the proposed adapted Sainte-Laguë methods are added to the existing results. The comparative analyses of monotone methods with divisor are also done. The obtained results would allow the argued choose of appropriate VD method. A case study in this aim is described.

There are also compared the theoretically obtained mathematical expressions on the average value of I_d index for optimal solutions using Hamilton method with results obtained by computer simulation.

The most known practices with refer to multi-optional decisions are, probably, the ones related to elections. Therefore, further, the addressed aspects of multi-optional decision-making systems will be investigated (not harming the universality) through party-lists elections.

2 The optimization problem

Let [8]: M – number of seats in the elective body; n – number of parties that have reached or exceeded the representation threshold; V – total valid votes cast for the n parties; $d = M/V$ – influence power (rights) of each elector (decider); V_i, v_i – number and, respectively, percentage of valid votes cast for party i; x_i, m_i – number and, respectively, percentage of seats to be allocated to party i; I_d – value of ARD index. Here $V_1 + V_2 + V_3 + \ldots + V_n = V$.

Knowing quantities (integers): M; n; V_i, $i = \overline{1,n}$, it is required to determine the nonnegative values of unknowns x_i ($i = \overline{1,n}$) – integers, which would ensure the minimization of the index I_d value

$$I_d = \sum_{i=1}^{n} |v_i - m_i| \to \min \tag{1}$$

in compliance with the restriction

$$\sum_{i=1}^{n} x_i = M. \tag{2}$$

Problem (1)-(2) is of mathematical programming in integers. The minimum value I_d^* of ARD index is obtained using Hamilton method [1, 4].

3 Disproportionality of Hamilton method's solutions

In this section, results of mathematical expectancy $\bar{I}_d^*$ of I_d^* values, obtained by simulation, are compared with the theoretical approximate.

3.1 Theoretical mathematical expectancy of disproportionality

In comparative analyses, but also for various forecasts, the definition domain of I_d^* values is of interest. Knowledge of definition domain and, also, of mathematical expectancy $\bar{I}_d^*$ expands the information about the possible I_d^* values in concrete elections in the tendency to minimize the index I_d value. This domain is determined in [9]. The knowledge of mathematical expectancy $\bar{I}_d^*$ of I_d^* values, theoretically investigated in [10] would also extend the possibilities of the mentioned above analyses.

Using I_d as index of disproportionality, in [10] it is obtained the following analytical expression for I_d^* for a specific election

$$I_d^* = \frac{200}{V}\sum_{j=1}^{\Delta M}(Q - R_j) = 200\left(\frac{\Delta M}{M} - \frac{1}{V}\sum_{j=1}^{\Delta M} R_j\right), \tag{3}$$

where $Q = V/M$ is the standard quota (Hare quota), and R_j, $j = \overline{1, \Delta M}$ are the largest ΔM remainders from the $\Delta V_i = V_i - a_i Q$, $i = \overline{1, n}$, $a_i = \lfloor V_i/Q \rfloor$ and

$$\Delta M = \frac{1}{Q}\sum_{i=1}^{n} \Delta V_i. \tag{4}$$

From (3), one can see that I_d^* depends on difference between ratios $\Delta M/M$ and $(R_1 + R_2 + \ldots + R_{\Delta M})/V$.

Mathematical expectancy $\bar{I}_d^*$ depends both, on the specificity of optimization problem, reflected in solution (3), and on the characteristics of the set of ballots for which it is determined. In [10] four approaches are proposed: 1) direct; 2) simplistic, based on the definition of I_d^*; 3) highly simplified, based on a conventional election, the characteristics of which are equal to certain average characteristics of an infinity of polls; 4) simplified, based on $n-1$ conventional polls, the characteristics of which are equal to certain average characteristics of an infinity of polls. In the following we will examine the first three approaches.

Direct approach involves the calculation of I_{dk}^* value of I_d^* index for each election k, and then the average $\bar{I}_d^*$ value on all K polls. At $K \to \infty$, $\bar{I}_d^*$ becomes mathematical expectancy. The main drawback of this approach is the difficulty of obtaining the analytical solution. The solution can be obtained only by simulation, some of results being described in s. 3.2.

The analytical solution can be obtained in the other three approaches: simplistic, highly simplified and simplified.

Simplistic approach assumes that the distribution of index I_d^* values is a symmetrical one to the middle of its definition domain $[\smile{I}_d^*, \frown{I}_d^*]$, and the $\bar{I}_d^*$ value may be determined as $\bar{I}_d^* = (\smile{I}_d^* + \frown{I}_d^*)/2 = \frown{I}_d^*/2$. Here $\smile{I}_d^* = 0$ is the lower limit and $\frown{I}_d^*$ – the upper limit of the definition domain. According to [10], in case of this approach $\bar{I}_d^*$ is calculated as follow

$$\bar{I}_d^* = \frac{\widehat{I}_d^*}{2} = \frac{25}{M}\begin{cases} n, \text{ at } n \text{ even} \\ n - \frac{1}{n}, \text{ at } n \text{ odd} \end{cases}, \quad \% \text{ of seats.} \tag{5}$$

From (5) it results that function $\bar{I}_d^*(M,n)$ is monotonically decreasing to M and monotonically increasing to n, and at even values of n and $M = n$ it does not depend on M and n. The upper limit of $\bar{I}_d^*(M,n)$, taking into account that $n \leq M$, is obtained at $M = n$: at even values of n, it does not depend on $M = n$ and is equal to 25%; at odd values of n, it increases with increasing of $M = n$ (since $sign(\partial \bar{I}_d^*/\partial n) = sign(n) > 0$ at $n \geq 3$) from $200/9\% \approx 22,22\%$, for $M = n = 3$, and tending to 25% for $M = n \to \infty$.

The highly simplified approach involves the use, as average value of I_d^* for an infinite number of ballots K, of the value $\tilde{I}_d^*$ for the conventional election with average remainders $\tilde{R}_j$, $j = \overline{1,n}$. According to [10], in case of this approach $\tilde{I}_d^*$ is determined as follows

$$\tilde{I}_d^* = \frac{25}{M}\begin{cases} n, \text{ at } n \text{ even} \\ (n+1)(1-\frac{1}{n^2}), \text{ at } n \text{ odd.} \end{cases} \tag{6}$$

From (6) it can be easily seen that function $\tilde{I}_d^*(M,n)$ is monotonically decreasing to M and monotonically increasing to n, and at even values of n and $M = n$ it does not depend on M and n. The upper limit of $\tilde{I}_d^*(M,n)$, taking into account that $n \leq M$, is obtained at $M = n$: at even values of n, it does not depend on $M = n$ and is equal to 25%; at odd values of n, it increases with the decreasing of $M = n$ (since $sign(\partial \tilde{I}_d^*(M,n)/\partial n) = sign(-n^2 + 2n + 3) < 0$ at $n > 3$) from $200/9\% \approx 22{,}22\%$, for $M = n = \infty$, and tending to $800/27\% \approx 29{,}63\%$ at $M = n = 3$.

Comparing expressions (5) and (6), it can be seen that for even n they coincide. Moreover, at $n = 2$, because always $\Delta M = 1$, these expressions convey exactly the average I_d^* value, i.e.

$$\left.\tilde{I}_d^*\right|_{n=2} = \frac{50}{M}\%. \tag{7}$$

3.2 Mathematical expectancy of disproportionality, by simulation

SIMOD application performs the direct approach for determining the disproportionality of seats allocation, according to Hamilton method, by computer simulation. The methodology for multi-optional PR voting systems computer simulation is described in [6]. Subject to simulation is only quantities V_i, $i = \overline{1,n}$.

The simulation was carried out, using the following initial data: N= 200000 (sample); $V = 100000000$; $M = 5, 10, 20, 50, 100$; $n = 2, 3, 4, 5, 7, 10, 15, 20, 50, 100$, $n \leq M$. When generating quantities V_i, $i = \overline{1,n}$, the uniform distribution was used. Some of the results of calculations are shown in Table 1.

Table 1. Average value $\bar{I}^*_{ds}$ of I^*_d, obtained by simulation, %

Seats,	Number of parties, n									
M	2	3	4	5	7	10	15	20	50	100
5	10,148	15,526	21,250	26,347						
10	4,980	7,790	10,419	13,042	18,026	26,350				
20	2,498	3,890	5,210	6,505	9,056	12,827	19,021	25,984		
50	0,998	1,555	2,086	2,600	3,620	5,135	7,646	10,148	25,504	
100	0,500	0,778	1,042	1,300	1,811	2,569	3,823	5,073	12,651	25,277

From Table 1 it can be seen that the $\bar{I}^*_{ds}$ value is decreasing both, to the number of seats M and to the number of parties n, the maximum value (about 25-26%) being reached at $M = n$. In practice, as a rule, cases for which $M = n$ are not met. Ratio n/M does not exceed, usually, 0,1 and then, as it is shown in Table 1, $\bar{I}^*_{ds} \leq 3\%$. Complementary to data of Table 1, for $M = 200$ and $n = 20$, it was obtained the value $\bar{I}^*_{ds} = 2,537\%$, down from $\bar{I}^*_{ds} = 5,073\%$ for $M = 100$ and $n = 20$. Data from Table 1 at $n = 2$ also confirm justice of estimate (7).

3.3 Comparative analyses

For comparative analysis of the three approaches: direct, simplistic and highly simplified, the essence of which is described in s. 3.1, in sections (a) and (b) of Table 2 there are shown respective quantitative values for the same sets of initial data values as those used for Table 1 in s. 3.2.

Table 2. Average value of I_d^* index conform to (5), (6) and combined

Seats, M	Number of parties, n: 2	3	4	5	7	10	15	20	50	100
	a) Simplistic approach ($\bar{I}_d^*$), %									
5	10	13,333	20	24						
10	5	6,667	10	12	17,143	25				
20	2,5	3,333	5	6	8,571	12,5	18,667	25		
50	1	1,333	2	2,4	3,429	5	7,467	10	25	
100	0,5	0,667	1	1,2	1,714	2,5	3,733	5	12,5	25
	b) Highly simplified approach ($\tilde{I}_d^*$), %									
5	10	17,778	20	28,8						
10	5	8,889	10	14,4	19,592	25				
20	2,5	4,444	5	7,2	9,796	12,5	19,911	25		
50	1	1,778	2	2,88	3,918	5	7,964	10	25	
100	0,5	0,889	1	1,44	1,959	2,5	3,982	5	12,5	25
	c) Combined approach ($\bar{I}_{dc}^*$), %									
5	10	15,555	21,094	26,4						
10	5	7,778	10,547	13,200	18,367	25,988				
20	2,5	3,889	5,273	6,600	9,184	12,994	19,289	25,561		
50	1	1,556	2,110	2,640	3,674	5,198	7,716	10,224	25,240	
100	0,5	0,778	1,055	1,320	1,837	2,599	3,858	5,112	12,620	25,123

Comparing data of sections (a) and (b) in Table 2 with those of Table 1, it can be seen that, for odd values of n, there are the following relations: $\bar{I}_d^* < \bar{I}_{ds}^* < \tilde{I}_d^*$, and for the even ones, except $n = 2$, on the contrary, $\bar{I}_{ds}^* > \bar{I}_d^* = \tilde{I}_d^*$. Therefore, section (c) of Table 2 presents data for the combined approach ($\bar{I}_{dc}^*$), the average ratio, in this case, being calculated according to formula

$$\bar{I}^*_{dc} = \frac{25}{M}\begin{cases} n, \text{ at } n = 2 \\ (n + \frac{1}{2})(1 - \frac{1}{n^2}), \text{ at } n > 2, \end{cases} \tag{8}$$

where $(n+1/2)(1-1/n^2) = [(n-1/n)+(n+1)(1-1/n^2)]/2$ from the second line of (5) and (6).

To compare the direct approach (by simulation) with theoretical approaches for each variant of the initial data, the absolute value of difference between the value of each element of Table 1 with that of the respective element of Table 2 is calculated. The obtained results are shown in Table 3.

Table 3. Value of difference between $\bar{I}^*_{ds}$ and indices values from Table 2

Seats,	Number of parties, n									
M	2	3	4	5	7	10	15	20	50	100
	a) Absolute value of difference $\bar{I}^*_{ds} - \bar{I}^*_{d}$, %									
5	0,148	2,192	1,250	2,347						
10	0,020	1,123	0,419	1,0420	0,883	1,351				
20	0,002	0,557	0,210	0,505	0,484	0,327	0,355	0,984		
50	0,002	0,222	0,086	0,200	0,191	0,135	0,179	0,148	0,504	
100	0,001	0,111	0,042	0,100	0,096	0,069	0,090	0,073	0,151	0,277
	b) Absolute value of difference $\bar{I}^*_{ds} - \tilde{I}^*_{d}$, %									
5	0,148	2,252	1,250	2,453						
10	0,020	1,099	0,419	1,358	1,566	1,351				
20	0,002	0,554	0,210	0,696	0,740	0,327	0,890	0,984		
50	0,002	0,223	0,086	0,280	0,299	0,135	0,319	0,148	0,504	
100	0,001	0,111	0,042	0,140	0,149	0,069	0,159	0,073	0,151	0,277
	c) Absolute value of difference $\bar{I}^*_{ds} - \bar{I}^*_{dc}$, %									
5	0,148	0,030	0,156	0,053						
10	0,020	0,012	0,128	0,158	0,341	0,363				
20	0,002	0,001	0,064	0,095	0,128	0,167	0,268	0,423		
50	0,002	0,001	0,024	0,040	0,054	0,063	0,070	0,076	0,264	
100	0,001	0,000	0,012	0,020	0,026	0,030	0,035	0,039	0,031	0,154

Comparing data of sections (a)-(c) of Table 3, it can be seen that the lowest absolute deviations from results, obtained by simulation, are for combined approach ($\bar{I}^*_{ds} - \bar{I}^*_{dc}$). Such deviations do not exceed, for

$M = n$, approx. 0,5%. At the same time, for cases encountered in practice (as a rule, $n/M \leq 0,1$), as shown in section (c) of Table 3, $|\bar{I}^*_{ds} - \bar{I}^*_{dc}| \leq 0,04\%$.

So, to forecasting the average disproportionality of seats allocation ($\bar{I}^*_d$), when applying the Hamilton method, it is appropriate to use the expression (8), the error not exceeding 0,5% of seats, and in most practical cases – 0,05% of seats.

4 Comparison of monotone VD methods by simulation

There are compared the well known d'Hondt, Sainte-Laguë and Huntington-Hill monotone VD methods. Some results of calculations at uniform distribution of quantities V_i, $i = \overline{1,n}$ and initial data: $M = 5, 10, 20, 50, 100$; $n = 3, 4, 5, 7, 10, 15, 20, 50$ ($n < M$); $V = 10^8$; sample size of 200000 ballots for each pair $\{M, n\}$, are presented in Figs. 1–3.

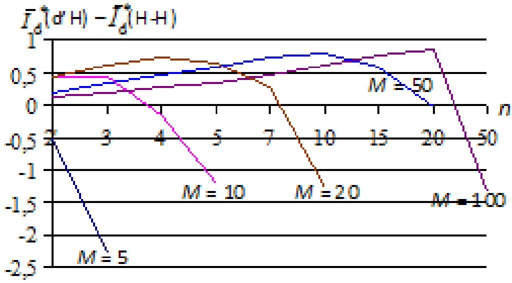

Figure 1. $\bar{I}^*_d$(dHondt) $-\bar{I}^*_d$(Huntington-Hill), %.

From Fig. 1 one can see that, by parameter $\bar{I}^*_d$, in some cases the Huntington-Hill method is better then the d'Hondt one and, in other cases, vice versa. Also, with the increase of M and decrease of n the Huntington-Hill method became better than the d'Hondt one. At the same time, for cases encountered in practice (usually, $n/M \leq 0,1$),

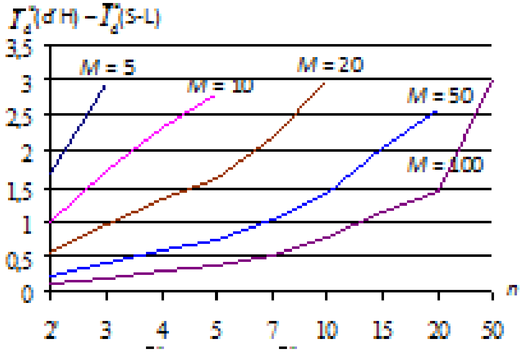

Figure 2. $\bar{I}_d^*$ (dHondt) $-\bar{I}_d^*$(Sainte-Lague), %.

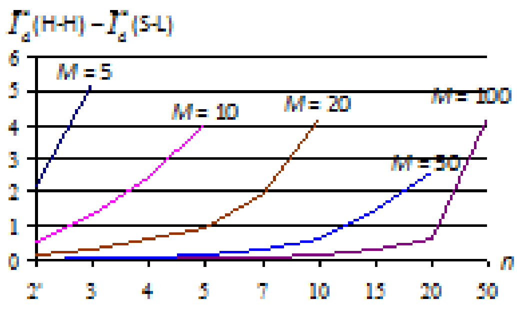

Figure 3. $\bar{I}_d^*$(Huntington-Hill) $-\bar{I}_d^*$(Sainte-Lague), %.

excepting $n = 2$ for small values of M, Huntington-Hill method is better than the d'Hondt one ($\bar{I}_d^*(d'H) - \bar{I}_d^*(H-H) > 0$).

Similarly, from Figures 2 and 3 one can see that Sainte-Laguë method is better then the d'Hondt and Huntington-Hill ones, no matter of parameters M and n values ($\bar{I}_d^*(d'H) - \bar{I}_d^*(S-L) > 0$, $\bar{I}_d^*(H-H) - \bar{I}_d^*(S-L) > 0$). Also, the value of differences $\bar{I}_d^*(d'H) - \bar{I}_d^*(S-L)$ and $\bar{I}_d^*(H-H) - \bar{I}_d^*(S-L)$ are increasing with the decrease of M and the increase of n.

So, the best, by parameter $\bar{I}_d^*$, from the examined monotone methods, is the Sainte-Laguë one. However, there may be particular cases, when the Huntington-Hill method, as well as the d'Hondt method, ensures a lower value of parameter I_d^* then the Sainte-Laguë one. To characterize such situations, parameters R_{SL-dH} and R_{SL-HH} are used. Parameter R_{SL-dH} is the ratio of the percentage of ballots, for which $\bar{I}_d^*$ (d'Hondt) $>$ $\bar{I}_d^*$(Sainte-Laguë), to the percentage of ballots, for which $\bar{I}_d^*$(d'Hondt) $<$ $\bar{I}_d^*$ (Sainte-Laguë). Similarly, R_{SL-HH} is the ratio of the percentage of ballots, for which $\bar{I}_d^*$(Huntington-Hill) $>$ $\bar{I}_d^*$(Sainte-Laguë), to the percentage of ballots, for which $\bar{I}_d^*$(Huntington-Hill) $<$ $\bar{I}_d^*$(Sainte-Laguë).

Some results of parameters R_{SL-dH} and R_{SL-HH} calculations, at uniform distribution of quantities V_i, $i = \overline{1,n}$ and initial data: $M = 20, 100$; $n = 3, 4, 5, 10$; $V = 10^8$; sample size of 200000 ballots for each pair $\{M, n\}$, are systemized in Table 1. Here P is the percentage of ballots, for which $\bar{I}_d^*$(Sainte-Laguë) $=$ $\bar{I}_d^*$(Hamilton)

Table 4. Some results of parameters P, R_{SL-dH} and R_{SL-HH} calculation

	$M = 20$				$M = 100$			
	n =3	n =4	n =5	n =10	n =3	n =4	n =5	n =10
P, %	94,04	91,28	89,03	85,63	94,02	91,43	89,09	81,60
R_{SL-dH}, times	12,84	13,46	15,57	32,11	12,82	14,41	16,59	37,60
R_{SL-HH}, times	4,79	5,71	6,84	25,49	2,68	3,19	3,65	5,38

From Table 1 one can see that, in cases of examined initial data, for more than 80% of ballots Sainte-Laguë method gives the same allocation of seats as the Hamilton one does ($\bar{I}_d^*$(Sainte-Laguë) = $\bar{I}_d^*$(Hamilton)). Also, Sainte-Laguë method gives a better distribution of seats for a number of polls at least 12-38 times higher than the d'Hondt one and at least of 2,525 times higher than the Huntington-Hill one does. Elsewhere, P index is decreasing and R index is increasing with the increase of the number n of parties: more parties – less efficient is Sainte-Laguë method in comparison with the d'Hondt and Huntington-Hill ones.

More than that, the Sainte-Laguë method meets the lower quota ($x_i \geq a_i$, $i = \overline{1,n}$), while the Huntington-Hill one may not satisfy it [3]. Another advantage of Sainte-Laguë method, at $n = 2$ it coincides with the Hamilton (optimal) one, while the d'Hondt and Huntington-Hill may not coincide [5]. Also, at $n = 2$ and $n = 3$, the Sainte-Laguë method meets the quota rule ($a_i \leq x_i \leq a_i + 1$, $i = \overline{1,n}$), while the d'Hondt and Huntington-Hill ones may not satisfy it [7]

Additionally, as it is shown in [11], Webster method is not always equivalent, as affirmed in [1] and other publications, to the Sainte-Laguë one. For equivalence, Webster method needs, in some cases, additional operations shown in [11]. As a result, the use of Sainte-Laguë method is easier than that of such complemented Webster method.

Overall, the use of Sainte-Laguë method is more efficient than that of d'Hondt and Huntington-Hill ones and is easier than that of complemented Webster method.

5 The adapted Sainte-Laguë method

In some cases it is needed to allocate to each party a number of seats not lower than an established value. For example, in the United States Congress House of Representatives each state shall have at least one representative (seat). The ordinary Sainte-Laguë method does not ensure such allocation of seats. But it is ensured by Huntington-Hill method and the described in this section adapted Sainte-Laguë (ASL) method.

The adapted Sainte-Laguë method differs from the Sainte-Laguë one only by satisfying the condition that each state shall have at least one representative (seat) in the House. According to this method, the allocation of seats to states shall be done as follows:

1. Let Ω be the set of states, $|\Omega| = n$. Allocating of seats to states with the number of population that do not exceed the standard quota (Hare quota) $Q = V/M$: for $i = \overline{1,n}$, if $V_i \leq Q$, then $x_i := 1$, $V := V - V_i$, $M := M - 1$, $\Omega := \Omega/i$

2. For the new set Ω of states and new values of parameters $n = |\Omega|$, M, V and $Q = V/M$, to allocate seats according to the ordinary Sainte-Laguë method. Stop.

Statement 5.1. The adapted Sainte-Laguë method is immune to the Alabama, of Population and of New State paradoxes.

Proof. With refer to Alabama paradox, we have: $M' > M$; $V_i' = V_i$, $i = \overline{1,n}$. Therefore, $V' = V$, $V'/M' = Q' < Q = V/M$. Here by stroke (‘) the parameters for the second ballot are noted. To avoid the Alabama paradox, it shall be $x_i' \geq x_i$, $i = \overline{1,n}$. Indeed, let $F_j = V_j/[2(x_j - 1) + 1] = \min\{V_i/[2(x_i - 1) + 1], i \in \Omega/G\}$, where G is the set of parties for which $V_i \leq$ and $x_i = 1$. Because of $Q' < Q$ and $V_i' = V_i$, $i = \overline{1,n}$, the number of parties for which $V_i' \leq Q'$ is not larger than that of set G. So, for states of set G, having $x_i = 1$, the condition $x_i' \geq x_i$ is satisfied. From the other hand, because of $M' > M$, it takes place $F_k' \leq F_j$, where $F_k' = \min\{V_i'/[2(x_i' - 1) + 1]\}$, $i \in \Omega/G$. That is why for states of set Ω/G the condition $x_i' \geq x_i$ is satisfied, too.

For the case of Population paradox, we have: $M' = M$; $V_i' = V_i$, $i = \overline{1,n}/k$; $V_k' > V_k$. Therefore, $V' = V + V_k' - V_k$, $V'/M' = Q' > Q = V/M$. To avoid the Population paradox, it shall be $x_k' \geq x_k$. Indeed, if $V_k \leq Q$, then $x_k = 1$ and because of restriction $(x_k, x_k') \geq 1$ anyway occurs $x_k' \geq x_k$. Let $V_k > Q$, then $V_k' > Q'$, too, because of $Q' = (V + V_k' - V_k)/M$ and $V_k' = V_k + (V_k' - V_k)$, from where $V_k' - Q' = V_k - Q + (V_k' - V_k)(1 - 1/M) > 0$, taking into account that $V_k' > V_k$ and $M > 1$. If so, and taking into account that $Q' > Q$, the number of parties for which $V_i' > Q$’ is not larger than that of set Ω/G. Therefore, and taking into account that $V_k' > V_k$, we have $F_k' > F_j$, where $F_k' = V_k'/[2(x_k' - 1) + 1]$ and $F_j = V_j/[2(x_j - 1) + 1] = \min\{V_i/[2(x_i - 1) + 1], i \in \Omega/G\}$ So, in case of $V_k > Q$ the condition

$x'_k \geq x_k$ is satisfied, too.

Finally, regarding the New State paradox, we have: $V'_i = V_i$, $i = \overline{1,n}$; $V_{n+1} = 0$; $V'_{n+1} > 0$; $V' = V + V'_{n+1}$; $M' = M + x_{n+1}$. The value of x_{n+1} is obtained by applying the respective optimization method. To avoid the New State paradox, it shall be $x'_i = x_i$, $i = \overline{1,n}$. Indeed, depending on relation between Q and Q' it may be, for the second ballot, that some states from set G will move to set Ω'/G' (if $Q > Q'$) or on the contrary some states from set Ω/G will move to set G' (if $Q < Q'$), or $G' = G \cup (n+1)$ (if $Q = Q'$) and no states move between these sets. But in no cases these movements of states between mentioned above sets do not influence the functions of preference of parties $F_i = V_i/[2(x_i - 1) + 1]$, $i = \overline{1,n}$ and therefore they do not change the allocation of seats to parties.

It is easy to observe that the adapted Sainte-Laguë method is immune to the Alabama, of Population and of New State paradoxes also in cases when it is needed to allocate to each party a number of seats not lower than an arbitrary nonnegative value, including larger than one.

6 A case study

From 1941, the allocation of seats for the United States Congress House of Representatives (the apportionment) is done using the Huntington-Hill method. Let us compare, the Huntington-Hill and the described in section 5 adapted Sainte-Laguë methods, when applied for apportionment.

In Table 5 there are systemized data of apportionment for US Census population in the period of 1940-2010 years and year 2014, being used the following notations:

ΔX_{HH} – the number of seats by which the Huntington-Hill apportionment differs from the optimal Hamilton one;

ΔX_{ASL} – the number of seats by which the adapted Sainte-Laguë apportionment differs from the optimal Hamilton one.

From Table 5 one can see that only in one from nine cases of apportionment, the adapted Sainte-Laguë method gets a less proportional

Table 5. Data of apportionment by Huntington-Hill and ASL methods

<table>
<tr><th rowspan="2"></th><th rowspan="2">ΔX_{HH}</th><th rowspan="2">ΔX_{ASL}</th><th rowspan="2">$\Delta X_{HH} - \Delta X_{ASL}$</th><th colspan="2">States for which x_i differs, and its deviation from the optimal value, obtained by Hamilton method</th></tr>
<tr><th>By Huntington-Hill method</th><th>By adapted Sainte-Laguë method</th></tr>
<tr><td>1940</td><td>2</td><td>2</td><td>0</td><td>Arkansas: -1, Nevada: +1</td><td>Arkansas: -1, Nevada: +1</td></tr>
<tr><td>1950</td><td>6</td><td>4</td><td>2</td><td>California: -1, Massachusetts: -1, Arkansas: -1, Hawaii: +1, Nevada: +1, Alaska: +1</td><td>California: -1, Arkansas: -1, Nevada: +1, Alaska: +1</td></tr>
<tr><td>1960</td><td>4</td><td>0</td><td>4</td><td>Illinois: -1, Massachusetts: -1, West Virginia: +1, New Hampshire: +1</td><td>None</td></tr>
<tr><td>1970</td><td>4</td><td>0</td><td>4</td><td>Illinois: -1, North Carolina: -1, Idaho: +1, Montana: +1</td><td>None</td></tr>
<tr><td>1980</td><td>4</td><td>2</td><td>2</td><td>California: -1, Massachusetts: -1, New Mexico: +1, Montana: + 1</td><td>California: -1, Montana: +1</td></tr>
<tr><td>1990</td><td>4</td><td>2</td><td>2</td><td>New York: -1, New Jersey: -1, Oklahoma: +1, Mississippi: +1</td><td>New York: -1, Washington: +1</td></tr>
<tr><td>2000</td><td>2</td><td>0</td><td>2</td><td>North Carolina: -1, Utah: +1</td><td>None</td></tr>
<tr><td>2010</td><td></td><td>2</td><td>-2</td><td>None</td><td>North Carolina: +1, Rhode Island: -1</td></tr>
<tr><td>2014</td><td>2</td><td>0</td><td>2</td><td>Pennsylvania: -1, Rhode Island: +1</td><td>None</td></tr>
</table>

result (two seats) than the Huntington-Hill method does (year 2010), when the Huntington-Hill method gets a less proportional result (two or four seats) then the adapted Sainte-Laguë method does in seven cases (years 1950, 1960, 1970, 1980, 1990, 2000 and 2014). In four cases the apportionments obtained by adapted Sainte-Laguë method coincide with the optimal Hamilton (years 1960, 1970, 2000 and 2014), when the apportionment obtained by adapted Huntington-Hill method coincide with the optimal Hamilton only in one case (year 2010). In one case (year1940), the apportionments, obtained by both compared methods, coincide. So, these particular cases confirm the fact that the adapted Sainte-Laguë method is considerably better than the Huntington-Hill one.

Let's look beyond. We will compare the Huntington-Hill and the adapted Sainte-Laguë methods, by computer simulation using the SIMOD application, for the same US Census years and the same US summary states population (V), but for general uniform or standard normal distribution of states population V_i, $i = \overline{1,n}$. As comparison criterion we will use R_{ASL-HH} – the ratio of the percentage of ballots, for which $\bar{I}_d^*$ (adapted Sainte-Laguë) $<$ $\bar{I}_d^*$ (Huntington-Hill), to the percentage of ballots, for which $\bar{I}_d^*$ (Huntington-Hill) $<$ $\bar{I}_d^*$ (adapted Sainte-Laguë)

The computer simulation was carried out using samples of 200000 ballots each. So, for uniform distribution of states population we have R_{ASL-HH} (1940) $= 6,720$ times and for the other eight years – $R_{ASL-HH} \in [7,276; 7,535]$ times. The value for the year 1940 differs essentially from values for the other eight years because in that year there were only 48 states. For standard normal distribution of states population, we have $R_{ASL-HH}(1940) = 2,183$ times and for the other eight years – $R_{ASL-HH} \in [2,258; 2,306]$ times. This data also confirm that, for apportionment, the adapted Sainte-Laguë method is considerably better than the Huntington-Hill one.

It was also determined that, at uniform distribution of states population, apportionments with adapted Sainte-Laguë method have the average value of I_d^* index $\bar{I}_d^*(ASL, 1940) = 2,946$ % of seats and for the other eight years – $\bar{I}_d^*(ASL) \in [3,074; 3,076]$ % of seats. Similar, at standard normal distribution of states population, we have $\bar{I}_d^*(ASL, 1940) = 2,785$ % of seats and for the other seven Census years and year 2014 – $\bar{I}_d^*(ASL) \in [2,899; 2,900]$ % of seats.

7 Conclusions

Combining theoretical results and computer simulation, the formula is obtained for the estimation of average disproportion of seats allocation, using Hamilton method, for each particular value of the number M of seats and of the number n of parties, the error not exceeding 0,5% of seats, and in most practical cases – 0,05% of seats. This formula can be used to predict the disproportionality of multi-optional decisions by Hamilton method.

By proportionality of voters' will representation in the final multi-optional decision, from the three compared monotone VD methods with divisor – d'Hondt, Huntington-Hill and Sainte-Laguë, the best is the last one. For example, in cases of examined initial data ($20 \le M \le 100, 3 \le n \le 10$ and uniform distribution of quantities V_i, $i = \overline{1, n}$) the Sainte-Laguë method gives a better distribution of seats for a number of polls at least 12-38 times higher than the d'Hondt one and at least of 2,5-25 times higher than the Huntington-Hill one does. Also, the use of Sainte-Laguë method is easier than that of complemented Webster method.

The Sainte-Laguë method is adapted to the requirement that each party (state) shall have at least one seat (representative) in the elective body. It is proved that the adapted Sainte-Laguë method is immune to the Alabama, of Population and of New State paradoxes. By computer simulation it is shown that ASL method is considerably better, in sense of minimizing the disproportion of seats allocation, than the Huntington-Hill one. So, for the US Census summary states population (V) and uniform distribution of states population $V_i i = \overline{1, n}$, the ASL method gives, in average, a better allocation of seats for a number of polls of 6,720 times higher for the Census year 1940, and of 7,276-7,535 times higher than the Huntington-Hill one does, for the other seven Census years in the period of 1950-2010 years (1950, 1960, ...,

2010) and year 2014. In case of standard normal distribution of states population V_i, $i = \overline{1,n}$, the ASL method gives, in average, a better allocation of seats for a number of polls of 2,183 times higher for the Census year 1940, and of 2,258-2,306 times higher than the Huntington-Hill one does, for the other seven Census years in the period of 1950 – 2010 years (1950, 1960, ..., 2010) and year 2014. Apportionments with adapted Sainte-Laguë method have the average value of I_d^* index equal to approximately $2,9 - 3,0$ seats.

These average results were confirmed by conventional apportionment for the United States Congress House of Representatives for Census values of states population V_i, $i = \overline{1,n}$ in the examined period: only in one, from the nine cases of apportionment, the ASL method gets a less proportional result than Huntington-Hill method does (year 2010), while the Hunting-ton-Hill method gets a less proportional result than the ASL method does in seven cases (years 1950, 1960, 1970, 1980, 1990, 2000 and 2014).

So, from the point of view of US Constitution requirement of proportional representation of states in the United States Congress House of Representatives, it is considerably better to use for apportionment the adapted Sainte-Laguë method than the used from 1941 year Huntington-Hill method.

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