Computer Science Journal of Moldova, vol.**20**, no.2(59), 2012

Comparison of indices of disproportionality in PR systems

Ion Bolun

Abstract

Comparative analysis of 12 indices of disproportionality by such characteristics as: metric, definition domain, representation uniformity, invariance and utilization field, is done. As a result of comparison, the opportunity to use as index of disproportionality in elections the Average relative deviation one is argued. Graphic representation for the upper limit of optimal solutions' disproportionality, when using this index, is shown.

Keywords: collective decisions, elections, proportional representation, index of disproportionality, comparative analysis.

1 Introduction

When taking collective decisions, using voting systems with proportional representation (PR), to minimize the disproportion of deciders' will representation in the decision is required – disproportion caused by the character in integers of the number of deciders and that of alternative options. To estimate this disproportion, various indices were proposed, some of which are described in [1-8].

As it is shown in [9], minimizing the disproportionality, within the meaning of each of 11 such indices (Rae [3], Loosemore-Handby [1], Rose [5], Grofman [6], Lijphart [4], Gallagher [1], Square deviation [9], Sainte-Laguë [1], d'Hondt [9], Mean relative deviation [8] and Relative standard deviation [8]), is ensured, as appropriate, by one of three methods (votes-decision rules – VD): Hamilton [10], Sainte-Laguë [1] or d'Hondt [1]. But other methods are known, too, including that of

Huntington-Hill [10], Largest remainders with Droop quota [7], Largest remainders with Hagenbach-Bischoff quota [2], Largest remainders with Imperiali quota [2], etc.

In significant number of cases, solutions obtained with different methods do not coincide, which, in pursuing the same goals, shows possible reserves in the accuracy of used voting systems. To eliminate the uncertainty of applying different VD rules in specific cases, it is important to elucidate, first, the essence and mission of voting, then to select the appropriate index for assessing the disproportionality of deciders' will representation in the decision, further formulation of the respective optimization problem and, finally, defining the method for determining the expected optimal solution.

Thus, one of the important steps in implementing a voting system, adequate to voting mission in a particular case, is to select the relevant index of disproportionality. A successful selection requires as more complete as possible comparative analysis of known indices. Some such issues are addressed in [1, 4-6, 8]. In the following an attempt of comprehensive characterization of 12 indices of disproportionality is taken.

The most known practices with refer to the use of voting systems are, probably, the ones related to elections. Therefore, further, the addressed aspects of indices of disproportionality will be investigated, not harming the universality, through the party-lists (blocks, coalitions) PR elections – RPL. Also, it is considered that all voters have equal rights, i.e. all votes have the same weight. Results, obtained in such assumptions, can be, as a rule, relatively easy extended for elections with weighted votes. First, in Section 2 the general formulation of the problem of optimizing the distribution of seats between parties is given, later the essence of 12 indices of disproportionality (Section 3) is described and the criteria for the comparison of these indices (Section 4) are listed. Subsequently, in Sections 5–9, the concerned indices are characterized in accordance with the comparison criteria and, finally, in Section 10 the multidimensional comparison of the 12 investigated indices is performed.

2 The problem of distribution of seats in LPR elections

Identification of the place and elucidation of the role of indices of disproportionality in LPR elections can be made basing on formalizing the problem of seats distribution. Let ($i = \overline{1,n}$): M – total number of seats in the elective body; n – number of parties that have reached or exceeded the representation threshold; V – total valid votes cast for the n parties; V_i – total valid votes cast for party i; x_i – number of seats to be allocated to party i.

If index I is used as disproportionality criterion, then the problem of optimizing the distribution of seats among n parties can be formulated as follows [8]. Let quantities (natural numbers) M; n; V_i, $i = \overline{1,n}$ are known and

$$\sum_{i=1}^{n} V_i = V. \tag{1}$$

It is required to determine the values of unknowns x_i ($i = \overline{1,n}$) – integers, which would ensure the extreme value for I (minimum or maximum, depending on the essence of I)

$$I = f(M; n; V_i, x_i, i = \overline{1,n}) \rightarrow \text{extremum} \tag{2}$$

in compliance with restrictions:

$$M = \sum_{i=1}^{n} x_i; \tag{3}$$

$$x_i \geq 1, \quad i = \overline{1,n}. \tag{4}$$

Sometimes, problem (2)-(4) is necessary to complete with monotonicity constraint, formalized in [11] as: to ensure the non-decreasing character of functions $x_i(D_i)$, $i = \overline{1,n}$, where [8]

$$D_i = dV_i = MV_i/V, \quad i = \overline{1,n} \tag{5}$$

are the party i rights in the elective body, delegated by the V_i votes of the electors (the summary value of the V_i votes), and $d = M/V$ is the value of one vote.

If, for example, in (2) the Sainte-Laguë index [1] is used as index I, then

$$I_{\text{S-L}} = \sum_{i=1}^{n} \frac{1}{v_i}(v_i - m_i)^2 = 100 \cdot V \sum_{i=1}^{n} \frac{1}{V_i} \left(\frac{V_i}{V} - \frac{x_i}{M} \right)^2 \to \min, \qquad (6)$$

where $v_i = 100V_i/V$, $m_i = 100x_i/M$. It can be easily proved (using the Lagrange multipliers method) that, when quantities $x_i \geq 0$, $i = \overline{1, n}$ are real numbers, the problem {(2)-(3), (6)} solution is given by equalities

$$m_i = v_i, \quad i = \overline{1, n}. \qquad (7)$$

This solution coincides with the proportional representation. But in practical cases, quantities x_i ($i = \overline{1, n}$) are integers, the problems (2)-(4) and (2)-(5) being of mathematical programming in integers.

In real elections, the probability to satisfy the equalities (7), at quantities x_i ($i = \overline{1, n}$) being integers, is very small. Thus, in real LPR elections there is a certain disproportion of seats distribution among parties. In such cases, it is important to assess the disproportionality in question. For this purpose, diverse criteria are used, called disproportionality indices, 12 of which are described in Section 3.

3 Indices of disproportionality

M. Gallagher highlights [1] two broad categories of measures of disproportionality in LPR elections: 1) measures based on the absolute difference between the party's seats and votes; 2) measures focused on the ratio between a party's seats and its votes. In both these categories, primary in assessing the disproportionality are parties. In reality, however, primary are voters; voters should be represented equally in the elective body or, if it is not possible, with a smallest possible disproportion. Therefore, at the base of the index of disproportionality the value of each vote should stand – vote that reflects unequivocally the rights of each voter in the election. Namely, starting from the value of a vote $d = M/V$, in [8] for this purpose the index of Mean relative deviation is derived and proposed, which subsequently was converted to the form that plays parties representation.

Diverse indices of disproportionality are known and used. The essence of twelve of them is the following.

Rae index [3], noted here I_{Rae}, is proposed in 1967 and is determined as the mean absolute deviation of the percentage of votes from the percentage of seats per one party

$$I_{\text{Rae}} = \frac{1}{n} \sum_{i=1}^{n} |v_i - m_i|. \tag{8}$$

Loosemore-Handby index [1], noted here $I_{\text{L-H}}$, is proposed in 1971, can be interpreted as the total absolute deviation between m_i and v_i for parties with deficit of seats or the percentage of seats taken from some parties (parties that have become, under this operation, with deficit of seats – losing parties) and distributed to other parties (parties that have become, under this operation, with excess of seats – gaining parties) and is determined as

$$I_{\text{L-H}} = \frac{1}{2} \sum_{i=1}^{n} |v_i - m_i|. \tag{9}$$

Rose index of proportionality [5], noted here I_{R}, is proposed in 1998, being a normalized version of Loosemore-Handby index, in such a way that a value of 100% corresponds to proportional representation, and 0% – to the worst case, and is determined as

$$I_{\text{R}} = 100 - \frac{1}{2} \sum_{i=1}^{n} |v_i - m_i|. \tag{10}$$

Grofman index [6], noted here I_{Gr}, is proposed in 1985, differs from the Rae one only by the replacement of n by "effective number" of parties N, introduced by Laakso and Taagepera in [15].

$$I_{\mathrm{Gr}} = \frac{1}{N}\sum_{i=1}^{n}|v_i - m_i|, \tag{11}$$

where $N = 10^4 \Big/ \sum_{i=1}^{n} v_i^2$. The value of N can be calculated from m_i [15], too, but, taking into account that when determining m_i, $i = \overline{1,n}$ primary are quantities v_i, $i = \overline{1,n}$ (and not vice versa) and quantities m_i, $i = \overline{1,n}$ can be with deviation from proportional representation, in this paper the already specified expression will be used.

Lijphart index [4], noted here I_{L}, is proposed in 1994 and represents the maximal absolute deviation between m_i and v_i

$$I_{\mathrm{L}} = \max_{i=\overline{1,n}}|v_i - m_i|. \tag{12}$$

Gallagher index [1], noted here I_{Ga}, is proposed in 1991, differs from the Loosemore-Handby one by the representation of not absolute but of square total deviation between m_i and v_i for parties with deficit of seats, amplifying the weight of larger differences $|v_i - m_i|$ at the expense of weights of smaller differences $|v_i - m_i|$, and is determined as

$$I_{\mathrm{Ga}} = \sqrt{\frac{1}{2}\sum_{i=1}^{n}(v_i - m_i)^2}. \tag{13}$$

It is easily seen that Gallagher index differs a little from the **Least square index**, noted here I_{LSM}, largely used in practice in diverse domains to assess discrepancies

$$I_{\mathrm{LSM}} = \sqrt{\sum_{i=1}^{n}(v_i - m_i)^2}. \tag{14}$$

Sainte-Laguë index [2], noted here $I_{\mathrm{S\text{-}L}}$, is determined as

$$I_{\mathrm{S\text{-}L}} = \sum_{i=1}^{n}\frac{1}{v_i}(v_i - m_i)^2 = \sum_{i=1}^{n} v_i(1 - \frac{m_i}{v_i})^2. \tag{15}$$

D'Hondt index [9], noted here I_{H}, represents the maximal ratio between v_i and m_i (in [2] in this purpose the maximal ratio between m_i and v_i) is proposed:

$$I_{\mathrm{H}} = \min_{i=\overline{1,n}} \frac{v_i}{m_i}. \qquad (16)$$

Mean relative deviation index [8], noted here I_{d}, specifies the average relative error (deviation) on election of the representation in the elective body of electors' rights $d_i = V_i/V$, $i = \overline{1,n}$ from the mean value $d = M/V$ and is determined as

$$I_{\mathrm{d}} = \frac{\Delta d}{d}100 = \sum_{i=1}^{n} |v_i - m_i|, \text{ where } \Delta d = \frac{1}{V}\sum_{i=1}^{n} V_i\,|d_i - d|. \qquad (17)$$

Here $|d_i - d| = \Delta d_i$ specifies the absolute error of the representation in the x_i seats of the value d of rights of each elector that votes for party i, and Δd – the mean absolute error per election (totality of V voters) of the representation in the elective body of an elector rights of value d. The mean relative deviation $100 \cdot \Delta d/d$, measured in percent of Δd by d, is equivalent, as it is shown in [8], to the percent of seats by which the distribution $\{x_1, x_2, \ldots, x_n\}$ differs from the distribution, which assures the equal representation in the elective body of electors' rights (of value d for each).

Relative standard deviation index [8], noted here I_σ, specifies the relative standard deviation of the representation in the elective body of electors' rights $d_i = x_i/V_i$ from the mean value $d = M/V$ and is determined as

$$I_\sigma = \frac{\sigma}{d}100 = 10\sqrt{\sum_{i=1}^{n} \frac{1}{v_i}(v_i - m_i)^2},$$
$$\text{where } \sigma = \sqrt{\frac{1}{V}\sum_{i=1}^{n} V_i(d_i - d)^2}. \qquad (18)$$

Here σ is the standard deviation of the representation in the elective body of electors' rights d_i, $i = \overline{1,n}$ from the mean value d, by applying, for simplicity, the division to V and not to $V-1$, the value of V being relatively large. In (18) I_σ is measured in percent of σ related to d.

Quasi invariant index [14], noted here I_{inv}, specifies the average number of seats to one party, by which the distribution $\{x_1, x_2, \dots, x_n\}$ differs from the proportional one and is determined as

$$I_{\text{inv}} = \frac{M}{100n} \sum_{i=1}^{n} |v_i - m_i|, \quad \text{seats/party.} \tag{19}$$

4 Criteria for the comparison of indices

The multitude of used indices of disproportionality is caused by the diversity of both, votes-seats (VS) rules applied in LPR elections and that of goals pursued in research. There is not yet a universally accepted index of disproportionality. M. Gallagher considers [1] that at theoretical level the soundest is, probably, the Sainte-Laguë index. One of the most used in the last years is the Gallagher index. Therefore, an important issue is to select or define such an index that will reflect as appropriate the essence of LPR election.

Of course, a comparative analysis of the respective indices would facilitate the successful selection of appropriate index in a certain situation. At the same time, a comparative analysis of various indices is difficult, because of their different meaning, in many cases; the value of already defined index is measured, as a rule, in different units. Hence, it follows the need to take into account, when comparing indices, diverse significant aspects, criteria. In the following, comparison will be made, where possible, by such characteristics as: metric, the definition domain, uniformity of representation, invariance and the usable field, including the comparative analysis.

5 Metric of indices

As it can be seen, the first five of indices listed in Section 3 – Rae (8), Loosemore-Handby (9), Rose (10), Grofman (11), Lijphart (12) and that of Mean relative deviation (17) are based on the absolute deviation of the percentage of seats m_i from the percentage of votes v_i; the following four – Gallagher (13), Least square (14), Sainte-Laguë (15) and that of Relative standard deviation (18) are based on the square deviation of m_i from v_i, and that of D'Hondt (16) is based on the ratio between v_i and m_i. Regarding the measure unit of their value, this is for the index ($i = \overline{1,n}$):

- Rae – %/party, the percentage being in sense of the absolute summary deviation of m_i from v_i. Therefore, this can be interpreted as the average percentage to a party of seats, by which the distribution x_i, $i = \overline{1,n}$ differs from the proportional one, being measured in %seats/party;
- Loosemore-Handby – %, the percentage being in sense of the absolute summary deviation of $m_i/2$ from $v_i/2$. In other words, this can be interpreted as the summary percentage of seats taken from some parties (parties that have become, under this operation, with deficit of seats – losing parties) and distributed to other parties (parties that have become, under this operation, with excess of seats – gaining parties), being measured in %lost seats/party;
- Rose – %, the percentage being in sense of complementary to 100% of Loosemore-Handby index (index of proportionality), namely is measured in 100% – %lost seats;
- Grofman – %/effective party, the percentage being in sense of absolute summary deviation of m_i from v_i, namely is measured in %seats/effective party;
- Lijphart – %, the percentage being in sense of absolute maximum deviation of m_i from v_i, namely is measured in %seats;

- Mean relative deviation – %, the percentage being in sense of average relative error per election of the representation in the elective body of an elector rights or, that is the same, – in sense of summary absolute deviation of m_i from v_i, namely is measured in %seats;
- Sainte-Laguë – %, the percentage being in sense of the average, weighted by $1/v_i$, of squared deviation of m_i from v_i;
- Gallagher – %, the percentage being in sense of square deviation of m_i from v_i, devided by $\sqrt{2}$, namely is measured in %seats/$\sqrt{2}$;
- Least squares – %, the percentage being in sense of square deviation of m_i from v_i, namely is measured in %seats;
- Relative standard deviation – %, meaning the percentage represented by the standard deviation σ of an elector rights $d = M/V$;
- D'Hondt – non dimensional, what part of m_i constitutes v_i for the smallest such ratio. This index multiplied by 100 can be measured in %;
- Quasi invariant – seats/party, representing the average number of seats to a party, by which the distribution x_i, $i = \overline{1,n}$ differs from the proportional one.

Thus, for all twelve listed above indices the used metric differs. Even for indices measured in percent, the percentage has a different sense, being from different absolute quantities.

The following analytical relations among some of the 12 indices (see [9], and (17) and (19)) facilitate the comparative understanding of indices' metric:

$$I_{\text{L-H}} = I_{\text{Rae}} \cdot n/2 = I_{\text{Gr}} \cdot N/2 = 100 - I_{\text{R}} = I_{\text{d}}/2 = 50nI_{\text{inv}}/M, \quad (20)$$

$$I_{\text{Ga}} = I_{\text{LSM}}/\sqrt{2}, \quad (21)$$

$$I_\sigma = 10\sqrt{I_{\text{S-L}}} \quad (22)$$

and $I_{\text{d}} \leq I_\sigma$, the equality taking place only in cases when $|v_i - m_i| = const$, $i = \overline{1,n}$.

6 Definition domain of indices' values

The minimum value for Rae, Loosemore-Handby, Grofman, Lijphart, Gallagher, Least squares, Sainte-Laguë, Mean relative deviation, Relative standard deviation and the Quasi invariant indices is zero. The value zero is obtained if equalities (7) take place, and it corresponds to proportional representation. In contrast, the value of Rose index, in these conditions, is maximal and is equal to 100%. And the minimal one is 0 and corresponds to complete disproportion. The minimal value of d'Hondt index is equal to 0 and is obtained if at least one seat is assigned to a party, that has not acquired any vote – the representation is not proportional; if equalities (7) take place, then $I_{\text{H}} = 1$, these being the I_{H} upper limit value and corresponding to proportional representation. It can be easily found, also, that if the threshold for party representation is 0, the indices' maximum limit value is: Loosemore-Handby, Lijphart and Gallagher – 100 %; Rae – $200/n$ %/party; Least squares – $100\sqrt{2}$ %; Grofman – 200%/party; Mean relative deviation – 200%; Sainte-Laguë and Relative standard deviation – ∞ and Quasi invariant – $2M/n$. For example, the upper limit for I_σ index is obtained, according to (18) and taking into account that $d = M/V$, in case of upper limit of the standard deviation σ, so of upper limit of the quantity $d_i = x_i/V_i$, namely it is ∞.

As noted in [13], the upper limit greater than 100%, for five indices of disproportionality (Grofman – 200%/party, Least squares – $100\sqrt{2}$, Sainte-Laguë – ∞, Relative standard deviation – ∞ and Mean relative deviation – 200%), in case of their using in problems of minimizing the disproportionality, is less informative. For example, it cannot be considered real an election, in which seats will be distributed to a party that did not receive any vote. In such cases, it is reasonable to take into account the definition domain of index values for optimal solutions. For 12 such indices (indices (8)-(18), described in Section 3, and the General divisor index), the domain in question is given in [13] and, partially, in Table 1, and for the Quasi invariant index [14] this is [0; 0,5] seats/party.

From Table 1 it can be seen that the definition domain for optimal solutions is considerably narrower than in the general case, the upper limit for disproportionality indices not exceeding 50% (Grofman and Mean relative deviation), and the lower one for indices of proportionality is not less than 50% (d'Hondt). In decreasing order, follows the upper limit for indices of disproportionality: Relative standard deviation – $100/\sqrt{3}\% \approx 57{,}7\%$; Least squares – $25\sqrt{2}\% \approx 35{,}4\%$, Sainte-Laguë – $100/3 \approx 33,3\%$, and the other (Rae, Loosemore-Handby, Lijphart and Gallagher) – 25%. The lower limit for Rose index of proportionality is 75%.

7 Uniformity of voters' will representation

In terms of uniformity of factors $|m_i - v_i|$ or v_i/m_i, $i = \overline{1,n}$ contribution to indices value, the twelve investigated indices can be grouped into two categories: 1) uniforms; 2) non uniforms. To uniforms the following indices relate: Rae, Loosemore-Handby, Rose, Grofman, Mean relative deviation and Quasi invariant, and to the non uniforms – indices: Sainte-Laguë, Gallagher, Least squares, Relative standard deviation, Lijphart and d'Hondt.

The non-uniformity of Sainte-Laguë, Gallagher, Least squares and Relative standard deviation indices is caused by the greater relative contribution to their value of greater deviations $|m_i - v_i|$ than that of smaller deviations $|m_i - v_i|$, because these deviations in indices are squared. Cause of the non-uniformity of Lijphart and d'Hondt indices is the contribution to their value of only one of the n factors (the largest absolute difference $|v_i - m_i|$ – in case of Lijphart index and the smallest ratio v_i/m_i – in case of d'Hondt index). By degree of uniformity, the non-uniform indices (of category 2) can be grouped into two subcategories: 2.1) partial uniforms – with a contribution other than 0 of all n factors (Sainte-Laguë, Gallagher, Least squares and Relative standard deviation); 2.2) non-uniforms – with the contribution of only one of the n factors (Lijphart and d'Hondt).

Table 1. Comparative characteristics of the 12 investigated indices

Index	Uniformity	Metric	Definition domain				Invariance to		Application field
			general case		optimal solution				
			min	max	min	max	M	n	
Indices of disproportionality									
Rae	uniform	%seats/party	0	100	0	25	no	yes	comparative analysis
Loosemore-Handby	uniform	%lost seats	0	100	0	25	no	no	universal
Grofman	uniform	%/ effective party	0	200	0	50	no	partial	comparative analysis
Lijphart	non uniform	%seats	0	100	0	25	no	no	universal
Gallagher	partial	%seats/$\sqrt{2}$	0	100	0	25	no	no	universal
Least squares	partial	%seats	0	$100\sqrt{2}$	0	$25\sqrt{2}$	no	no	universal
Sainte-Laguë	partial	%	0	∞	0	100/3	no	no	universal
Mean relative deviation	uniform	%seats	0	200	0	50	no	no	universal
Relative standard deviation	partial	%	0	∞	0	$100/\sqrt{3}$	no	no	universal
Quasi invariant	uniform	seats/party	0	$2M/n$	0	0,5	yes	yes	comparative analysis
Indices of proportionality									
Rose	uniform	100%-% lost seats	0	100	75	100	no	no	universal
D'Hondt	non uniform	%	0	100	50	100	no	no	universal

It may be said, broadly, that the subcategory 2.2 indices are more non-uniform, than those of subcategory 2.1.

Thus, in order of decreasing degree of uniformity of factors $|v_i - m_i|$ or those of v_i/m_i contribution to the value of indices, follow: uniform indices (Rae, Loosemore-Handby, Rose, Grofman, Mean relative deviation and Quasi invariant), then the partial uniform (Sainte-Laguë, Gallagher, Least squares and Relative standard deviation) and, finally, the non-uniform ones (Lijphart and d'Hondt).

8 Invariance to M, V and n

In comparative analysis of various cases of collective decision making by voting, indices, invariant to some initial data, are useful. In LPR voting systems a part or all quantities M, V and n can serve as such initial data, depending on the purpose.

It can be proved that at $V >> M \geq n$ all the 12 indices, described in Section 3, are, practically, invariant to the value of V. Also, Rae index (8) provides, to a considerable extent, invariance to the number n of parties, too, and the Grofman one (11) – to the effective number N of parties. Also, Quasi invariant index (19) ensures, to a considerable extent, both invariance to M and to n [14].

9 Application field

The appropriate field of using the 12 indices, described in Section 3, depends on their essence. At first analysis, one can conclude that the use of Rae (8), Grofman (11) and Quasi invariant (19) indices, for assessing the overall disproportionality of seats distribution in practical LPR elections, is not opportune. The first two of these three indices characterize the average disproportionality to a party (ordinary or "effective"), which is measured, respectively, in %seats/party and %seats/effective party. The third index also characterizes the average disproportionality to a party (ordinary), but is measured in seats/party. Therefore, these three indices is opportune to use in comparative analysis of disproportionality in diverse LPR elections, the Rae index assuring invariance to the number n of parties, the Grofman index – to the effective number N of parties, and the Quasi invariant index – both to n and to the overall number of seats M. At the same time, in such researches, the use of Rae index is preferable to that of Grofman index, the last weighting the n parties depending on values of quantities v_i, $i = \overline{1,n}$. The value of Grofman index increases with the increasing of dispersion of quantities v_i, $i = \overline{1,n}$.

The other nine indices, from the 12 described in Section 3, are designed to assess the overall disproportionality of seats distribution in LPR elections, although they may be used, in some cases with certain reservations, in comparative analysis, too. Since the Mean relative deviation index characterizes the error of elector's rights (vote) representation, this could be the most appropriate index for comparative analysis of various elections by the election overall disproportionality, too.

10 Multidimensional comparison of indices

The values of some comparison criteria for the 12 investigated indices are presented in Table 1. Preference for a particular index, based on these criteria, depends on the followed purpose. At such purposes may be referred: assessment of overall disproportionality per election; assessment of average disproportionality per election; assessment of maximal disproportionality per party; invariance to M, n and V, etc. In this section, only aspects referring to the assessment of overall disproportionality per election will be investigated, beginning with the correlation among the 12 investigated indices.

The correlation among indices, determined by relations (20)–(22), facilitates their comparative analysis. These relations cover 10 of the 12 investigated indices and define three groups of indices, strongly correlated with each other: a) $I_{\text{L-H}}$, I_{Rae}, I_{Gr}, I_{R}, I_{d}, I_{inv}; b) I_{Ga}, I_{LSM}; c) $I_{\text{S-L}}$, I_{σ}. Even if these indices differ by absolute value, all indices of each of these groups lead, in terms of minimizing the disproportionality, to the same solution: the fact of ensuring minimum of disproportionality (within the meaning of problem (2)–(4)) for an election under one of them ensures minimum of disproportionality under each of the other indices of this group. So, in terms of minimizing the disproportionality, any of indices of that group can be used as optimization criterion (2).

Moreover, using the two indices of each of groups (b) and (c) in comparative analysis of elections, also lead to similar results – for example, when ordering elections in increasing order of disproportionality. This statement occurs for indices $I_{\text{L-H}}$, I_{R} and I_{d} of group (a), too, but not occurs for the other three indices of this group. Index I_{Rae} (8) is highly sensitive to the number of parties, representing, in fact, the average disproportionality by a party and not by the election as a whole, even if with the increasing number of parties the probability of greater disproportionality increases too. Index I_{Gr} (11) is also dependent on the number of parties; trying to bring elections with different numbers of parties to a common denominator, by weighting parties by the number of votes cast, this complicates and, also, makes less clear the essence of evaluated disproportionality.

From (21) it results that Gallagher index differs from the Least squares one only by the constant $1/\sqrt{2}$. Although, by normalization (dividing by $\sqrt{2}$), Gallagher index assures the definition domain $[0, 100]\%$, it becomes, at the same time, less clear the essence of the percentage of disproportionality. Moreover, the definition domain of optimal solutions, when using the Least squares index, is $[0; 25\sqrt{2}]\%$ (that for the general case is $[0; 100\sqrt{2}]\%$), the upper limit being much less than 100%. Therefore, from these two indices it would be preferable, however, the use of the Least squares index – a well-known and widely used in various fields index, the percentage being interpreted in the usual sense for this index.

In a similar mode, from (22) it results that using of both indices – Sainte-Laguë and Relative standard deviation one for assessing the disproportionality, is useless. The use of indices $I_{\text{S-L}}$ and I_{σ} as optimization criterion (1) as well as in comparative analysis of LPR elections leads to similar results. For both these indices, the definition domain in general case is $[0; \infty]$, even in case of the optimal solutions this is $[0; 100/3]$ for the Sainte-Laguë index and $[0; 100/\sqrt{3}]$ – for the Relative standard deviation index. However, the essence of index I_σ is easier to understand, the standard deviation being universally accepted and widely used in various fields, and the fact, that it is taking into account the relative standard deviation, do not worsen the situation. Thus, from these two indices it would be preferable, however, the application of Relative standard deviation index.

According to equalities (20), there exist similarities, with accuracy to constants, among indices Loosemore-Handby, Rose and the Mean relative deviation one. With refer to the other three indices from (20), namely Rae, Grofman and Quasi invariant ones, in their calculation the quantities n, N and M are also used, depending on the case.

The preliminary selection of an index for assessing the overall disproportionality per election. By uniformity, the uniform indices are preferable (with uniform contribution of factors $|m_i - v_i|$ or v_i/m_i, $i = \overline{1,n}$ to indices' value), the partially uniform ones being less preferred, and the non-uniform – non-preferred. In terms of metric, it would be preferable an index with a clear, successful interpretation of the measure unit in investigated purpose; it should adequately reflect the essence of LPR elections. Also, it would be better if the definition domain of the selected index is $[0, 100]\%$. Invariance, in examined case, does not matter, but preferably it should be an index with universal field of use. Obviously, you might not find an index that meets all these requirements. Then, it will be necessary to take into account the preferences of comparison criteria of greater importance.

In terms of uniformity of the representation of voters' will in the elective body, it would be preferable using, for this purpose, one of the uniform indices, namely: Rae, Loosemore-Handby, Grofman, Mean relative deviation, Quasi invariant or Rose ones. Indices Lijphart (12) and d'Hondt (16) are non-uniforms ones, not covering the entire set V of votes of the election, but only the votes of electors, supporters of the party with the highest deviation $|v_i - m_i|$ (Lijphart index) or, respectively, with the highest ratio v_i/m_i (d'Hondt index). Thus, these indices, being, of course, useful for specific research, less can be used to characterize the overall disproportionality of an LPR election. With refer to indices of Least squares and the Relative standard deviation, they, like the Sainte-Laguë and Gallagher ones, are characterized by a higher relative contribution to their value of larger deviations $|m_i - v_i|$ comparatively to the relative contribution of smaller deviations $|m_i - v_i|$ (because these deviations in the index are squared); so, by definition, they cannot properly appreciate the proportional representation of voters' will of value d in the elective body. Therefore, their application, for investigated case of equality of electors' votes, is not so welcome. Their use may be indicated only in cases, when the application of uniform indices makes impossible the necessary research, the main drawback of assessing the disproportionality by absolute deviations $|v_i - m_i|$, as in indices (8) – (12), (17) and (19), consisting in that that the respective functions are not differentiable, which makes, in some cases, more difficult their apply in practice.

From the six indices, remaining to compare, the Rae, Grofman and Quasi invariant indices, according the invariance (and metric, too), are not appropriate, because they characterize the average disproportionality to a party and not that to the entire election. So we have to select one of the following three indices: Loosemore-Handby, Mean relative deviation and Rose ones. The Loosemore-Handby and Rose indices, in terms of using as index of disproportionality/proportionality across elections, are, as it is easily seen from their essence, mutually complementary and equivalent by efficiency. The difference is only in that that one (Loosemore-Handby) characterizes disproportionality, and the other (Rose) – "the degree of proportionality" of the representation in the elective body of electors' votes. Thus, for comparison with that of Mean relative deviation, one of these two indices is sufficient. However, with refer to the proportionality, it either exists (disproportionality is equal to 0) or does not exist (disproportionality is different from 0); so it is less successful to estimate "the degree of proportionality". Thus, the comparison will be provided with Loosemore-Handby index.

The Loosemore-Handby and Mean relative deviation indices differ only by constant 1/2. Their use both as optimization criterion (2), as well as in comparative analysis of LPR elections lead to similar results. Apparently, a slight advantage of the Loosemore-Handby index could be in that that it is normalized, ensuring the definition domain $[0;100]\%$, while that of the Mean relative deviation is $[0,200]\%$ (see table 1). But, at a deeper analysis, this advantage becomes disputable, the normalization of Loosemore-Handby index leads to loss of essence in the content of the measure unit of disproportionality.

Indeed, with refer to the metric, from the 12 indices, including the Loosemore-Handby one, the clearest interpretation, in sense of conditions of proportionality (7), has the Mean relative deviation index. It measures the average relative error per election of the representation in the elective body of an elector rights of value $d = M/V$, equal to the summary percentage of seats, by which the distribution x_i, $i = \overline{1,n}$ differs from the proportional one. While, taking into account that the percentage of over represented votes is equal to that of under-represented votes in the elective body, the Loosemore-Handby index measures the percentage of additional seats allocated to votes (parties) over represented in the elective body or, equivalently, the percentage of more seats that should have to be allocated to votes (parties) under-represented in the elective body. Given the negative sense of disproportionality, the accent should be opportune to return to the second interpretation. However, if taken into account only this aspect, rendering of disproportionality would be not correct, because not only a part of votes is represented with lack of seats; the situation gets worse, in the same way, by that that another part of votes is represented with a surplus of seats. Thus, Loosemore-Handby index yields, with refer to the adequate reflecting of disproportionality, to that of Mean relative deviation.

Let's concretize this loss of essence basing on some examples. For this purpose, let's elucidate, first, the conditions, in which the Mean relative deviation index reaches 200% seats. This happens, for example, when, from two parties participating in elections, to party, which received all the votes, not to award any vote, although at proportional representation it should be allocated to it 100% of votes (deviation of 100% from the proportional distribution), and to party, which did not receive any vote to allocate all 100% of votes (deviation from the proportional distribution of another 100%). Thus, the error of 100%

distribution of seats is doubled.

Another example. Let it be two localities A and B, forming a territorial unit with a common budget, managed by a Council elected by vote. The two localities are represented in the Council by a number of councilors, respectively x_A and x_B, according to the number of inhabitants, respectively V_A and V_B (all inhabitants of the two localities have the right to vote and voted, arguing the locality representatives, and all ballots were valid). To organize the Christmas holidays, the Council decided to grant the two localities by one financial support F_A and F_B, proportional to the number of councilors x_A and x_B, i.e.: $F_A = (F_A+F_B)x_A/(x_A+x_B)$ and $F_B = (F_A+F_B)x_B/(x_A+x_B)$. One asks: Loosermore-Handby index or that of Mean relative deviation more adequately reflects disproportionality in providing the financial support in question?

Let: $V = V_A + V_B$, $F = F_A + F_B$ and $M = x_A + x_B$. Obviously, granting financial support is proportional, if to any individual citizen, regardless of locality, is returned the same financial support equal to $F/V = d_F$. But in fact to a resident of locality A a support equal to F_A/V_A is returned, and to a resident of locality B – a support equal to F_B/V_B.

Thus, the absolute difference between each of ratios F_A/V_A and F_B/V_B and the ratio F/V is the absolute disproportionality per capita in their localities, and the mean relative disproportionality I is

$$\begin{aligned} I &= \frac{1}{d_F V}\left[V_A\left|\frac{F_A}{V_A}-\frac{F}{V}\right|+V_B\left|\frac{F_B}{V_B}-\frac{F}{V}\right|\right] = \\ &= \frac{1}{F}\left[V_A\left|\frac{Fx_A}{MV_A}-\frac{F}{V}\right|+V_B\left|\frac{Fx_B}{MV_B}-\frac{F}{V}\right|\right] = \\ &= \left[\left|\frac{x_A}{M}-\frac{V_A}{V}\right|+\left|\frac{x_B}{M}-\frac{V_B}{V}\right|\right], \end{aligned}$$

which, multiplied by 100%, coincides with the Mean relative deviation index for this case. Thus, the Mean relative deviation index more appropriately reflects the disproportionality in providing the nominee financial support.

To note, also, that in practice there are other cases, too, when the used index has a definition domain exceeding 100%, for example, inflation and profitability. However, regarding the definition domain for optimal solutions, it, in case of Mean relative deviation index, is [0, 50]% (Table 1), the upper limit $\widehat{I}_d^*$, calculated according to the expression from [8], is achieved in rare cases and being at least two times less than 100% (Fig. 1).

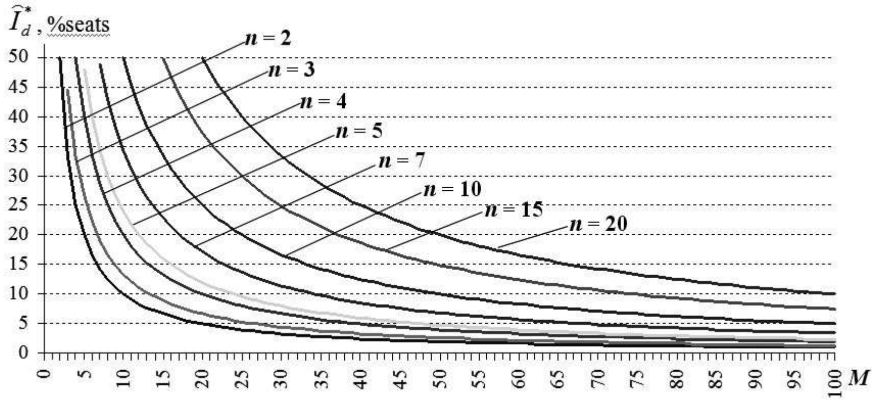

Figure 1. Graphics of the function $\widehat{I}_d^*(M, n)$

About some "shortcomings" of the Mean relative deviation index. The Mean relative deviation index is proposed recently in [8] and is therefore unknown. However, it is largely similar to the Loosemore-Handby one, well known and widely applied. The difference is only in constant 1/2 and the content essence of the measure unit of disproportionality, discussed above. Therefore, the shortcomings of the Loosemore-Handby index, except that regarding the content essence in question, are reflected, almost equally, to the Mean relative deviation index, too.

On vulnerability of the Mean relative deviation index to "paradoxes". A comparative analysis of six indices of disproportionality, basing on statistical data for 82 elections in 23 countries, is given in [1]. Regarding the Loosemore-Handby index, in [1], on the basis that its application as criterion of minimizing the disproportionality leads to the use of Largest remainders method as VS rule, it is concluded that the vulnerability of the method in question to the Alabama, the Population and the New State paradoxes (see, for example, [1, 11]) implies the vulnerability to these paradoxes of the index itself. To note that, at such approach, in the same situation as the Loosemore-Handby index are the Rae, Rose, Grofman, Lijphart, Gallagher, Least squares and Mean relative deviation indices, the use of which as optimization criterion (2) implies [9] the solving of problem (2)-(3) also by the Largest remainders method.

In reality, these are two different aspects: one and the same criterion can be used in various problems, which, in their turn, may involve different optimization methods. In this particular case, if one wants that the optimal solutions not to be vulnerable to mentioned above paradoxes, it is sufficient to complete the optimization problem (2) – (3) with the constraint of ensuring a non-descending character of functions $x_i(D_i)$, $i = \overline{1, n}$, taking into account the relation (5). Such a problem, in which as optimization criterion (2) the Loosemore-Handby index is applied, already cannot be solved using the method of Largest remainders. Expected solution can be obtained according to monotone method described in [11], when using as divisor the expression $ca_i + 1$, where $a_i = \lceil dV_i \rceil$, $i = \overline{1, n}$, and c is the average of ratio $\Delta M/n$ for the given voting system. Here $\Delta M = M - (a_1 + a_2 + \ldots + a_n)$. If ΔM has a symmetric distribution in the interval $[1; n-1]$ from the middle of this interval, it takes place $c = 2$ [11].

About the sensitivity of the Mean relative deviation index to the number of parties. In [16], the Loosemore-Handby index is criticized as one "much too sensitive" to the number of parties. However, in [1], on the contrary, it is said that it is "much too insensitive" to the number of parties, basing on the following example. Let it be two elections, (a) and (b), with the following features:

1. $v_1 = 60\%$ votes, $m_1 = 64\%$ seats; $v_2 = 40\%$ votes, $m_1 = 36\%$ seats; $I^a_{\text{L-H}} = 4\%$; $I^a_{\text{Rae}} = 4\%$;
2. $v_1 = v_2 = v_3 = v_4 = 15\%$ votes, $m_1 = m_2 = m_3 = m_4 = 16\%$

seats; $v_5 = v_6 = v_7 = v_8 = 10\%$ votes, $m_5 = m_6 = m_7 = m_8 = 9\%$ seats; $I^b_{\text{L-H}} = 4\%$; $I^b_{\text{Rae}} = 1\%$.

According to Loosemore-Handby index, in both elections there is the same disproportionality of 4%, while under the Rae one the disproportionality in election (a) is 4%, and in (b) – 1%, i.e. in election (b) the disproportionality is considerably lower, than in the (a) one.

Which of these two indices assessed fairly, in reality, the disproportionality in these two elections? If to consider that the Loosemore-Handby index appreciates the overall disproportionality per election, and the Rae – the average disproportionality to a party, then there is no contradiction here, except the absolute value of these two indices. But if the Rae index also to apply for assessing the overall disproportionality per election, than more accurate is the assessment under the Loosemore-Handby index, showing that the disproportionality in the two elections is the same. Indeed, in both elections, the percentage of seats, taken by parties with seats deficiency and distributed to parties with seats in excess, is the same. And if to assess these elections, using the Mean relative deviation index, then the average relative error per election of the representation of an elector rights of value $d = M/V$ in the elective body is the same in both elections, being equal to 8%.

So, the Alabama, the Population and the New State paradoxes are not related to Loosemore-Handby index and, also, one cannot say that it is "much too sensitive" or "much too insensitive" to the number of parties. This statement equally refers to the Mean relative deviation index.

The general scheme of preference of investigated indices, for assessing the total vote disproportionality, is given in Figure 2. From the discussion above, one can conclude that the most suitable index, for assessing the total vote disproportionality, including the comparative analysis of various elections, is the Mean relative deviation index.

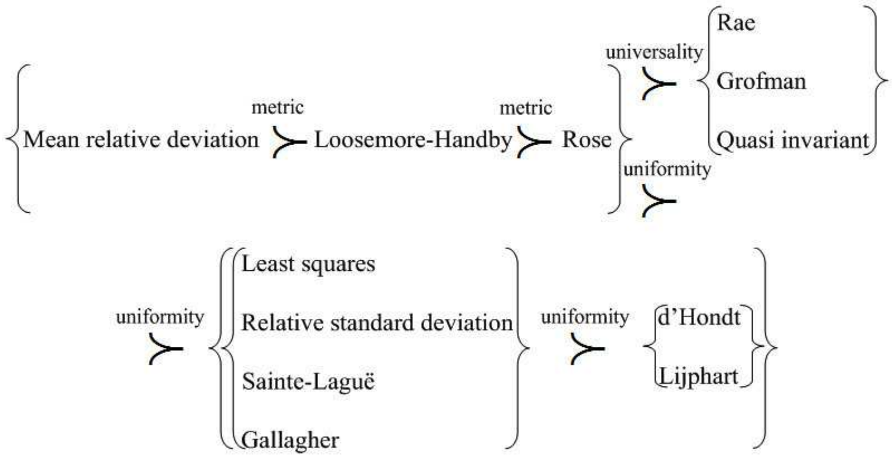

Figure 2. Scheme of preference of indices for assessing the total vote disproportionality

11 Conclusion

To identify the most appropriate index, for assessing the total vote disproportionality of the distribution of seats between parties in LPR systems, 12 indices are comparatively investigated. Comparison is made basing on the following features: metric, definition domain, uniformity of representation, invariance and field of use. Giving priority to the uniformity of voters' wills representation, and then, taking into account other characteristics, too, the set of candidate indices was reduced to three: Loosemore-Handby, Rose and the Mean relative deviation ones, then, finally, be argued the choice of the Mean relative deviation index. The last is uniform, with clear essence (the average relative error per election of the representation in the elective body of a voter's rights of value $d = M/V$), being measured in %seats, the definition domain for optimal solutions $[0, 50\%]$, dependent on M and n and can be used both for assessing of the total vote disproportionality and in comparative analysis of various elections. For a relatively large number of cases ($n \in [2; 20]$ and $M \in [2; 100]$), it is given the graphical representation of the upper limit of disproportionality of optimal solutions, when using as optimization criterion the Mean relative deviation.

References

[1] Gallagher M. *Proportionality, Disproportionality and Electoral Systems.* In: Electoral Studies (1991), 10:1, pp. 33–51.

[2] Sorescu A., Pîrvulescu C., et al. *Electoral systems.* – Bucureşti: ProDemocraţia, 2006. – 54 p. (in Romanian)

[3] Rae D.W. *The Political Consequences of Electoral Lows.* New Heaven: Yale University Press, 1967.

[4] Lijphart A. *Electoral Systems and Party Systems.* Oxford, Oxford University Press, 1994.

[5] Rose R., Munro N., Mackie T. *Elections in Central and Eastern Europe Since 1990.* Strathclyde: Center for the Study of Public Policy, 1998.

[6] Grofman B. and Lijphart A. *Electoral Lows and their Political Consequences.* New York: Agathon Press, 1986.

[7] Droop H.R. *On Methods of Electing Representatives.* In: Journal of the Statistical Society of London, Vol. 44, No. 2. (Jun., 1881), pp. 141–202 (Reprinted in Voting matters, No. 24 (Oct., 2007), p. 7–46).

[8] Bolun I. *Seats allocation in party-list election.* In: Economica, nr.2(76)/2011. – Chişinău: ASEM.

[9] Bolun I. *Algorithmization of optimal allocation of seats in PR systems.* In: Economica, nr.3(77)/2011. – Chişinău: ASEM. – pp. 137–152.

[10] Tannenbaum P. *Excursions in Modern Mathematics*, Seventh Edition. Pearson, 2008. – 704 p.

[11] Bolun I. *"Votes-decision" monotone method in PR systems.* In: Economica, nr.4(78)/2011. - Chişinău: ASEM. – pp. 108–117.

[12] Grofman B., Lijphart A. *The evolution of electoral and party systems in the Nordic countries.* – New York: Agathon Press, 2002. - 314 p.

[13] Bolun I. *Definition domain of optimal solutions disproportionality in PR systems.* In: Annals of Academy of Economic Studies of Moldova, Ed. 10. – Chişinău: ASEM, 2012. (in Romanian)

[14] Bolun I. *Quasi Invariant Index for the Estimation of Disproportionality in PR Systems.* In: Economica, nr.2(80)/2012. – Chişinău: ASEM.

[15] Laakso M., Taagepera R. *"Effective" Number of Parties: a Measure with Application to West Europe.* In: Comparative Political Studies, 12:1, 1979, pp. 3–27.

[16] Lijphart A. *The Field of Electoral Systems Research: A Critical Study.* In: Electoral Studies, 4:1, 1985, pp. 3–14.

Ion Bolun,

Received June 11, 2012

Academy of Economic Studies of Moldova
Bănulescu-Bodoni, 61, Chişinău, Moldova
Phone: +373 22402716
E–mail: *bolun@ase.md*