The geometry of fractional osculator bundle of higher order and applications

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Abstract

Using the reviewed Riemann-Liouville fractional derivative we define the bundle $\overset{\alpha k}{E} = Osc^{\alpha k}(M)$ and highlight geometrical structures with a geometrical character. Also, we introduce the fractional osculator Lagrange space of k order and the main structures on it. The results are applied at the k order fractional prolongation of Lagrange, Finsler and Riemann fractional structures.

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1 Introduction

It is known that the operators of integration and derivation have geometrical and physical interpretations and they were used in the modelation of problems from different domains. The use of reviewed Liouville-Riemann integration and derivation operators lead to fractional integration and derivation. The geometrical and physical interpretation is suggested by the Stieltjes integral and it was done by I. Podlubny [7]. There is a vast bibliography which contains the properties of fractional integral and derivative and the analysis of the processes which are modeled with their help [2], [5], [8].

A lot of models which use the fractional derivative are defined on an open set in $\mathbb{R}^n$. In this paper we present the fractional derivative taking into account the geometrical character, namely the behavior of associated objects under a change of local chart.

The outline of this paper is as follows. In Section 2 we describe the reviewed fractional derivative on $\mathbb{R}$ using [2], [5], the fractional osculator bundle $T^{\alpha}(M)$ on a differentiable manifold and the behavior of introduced objects under a change of local chart. In Section 3, we define the fractional osculator bundle of k order using the method presented by R. Miron in [6]. We introduce: the Liouvile fractional vector fields, the αk-fractional spray and the fractional nonlinear connection. We prove that these objects have a geometrical character. Our findings are analogous with R. Miron's results for the fractional case. In Section 4 we describe the fractional Euler-Lagrange equations for fractional osculator Lagrange spaces of superior order. The results are applied for the k-order fractional bundle prolongation of Lagrange, Finsler and Riemann structures.

The main results from the present paper were used in [3] and [4] for the study of some fractional geometrical structures and they will permit the study of other structures of this type.

2 The fractional derivative on R. The fractional osculator bundle on the differentiable manifold.

2.1 The fractional derivative on R

Let $f : [a, b] \to \mathbb{R}$ be a derivable function and $\alpha \in \mathbb{R}$, $\alpha > 0$. The functions:

$$
\begin{aligned}
{}_aD_t^{\alpha} f(t) &= \frac{1}{\Gamma(m-\alpha)}(\frac{d}{dt})^m \int_a^t (t-s)^{m-\alpha-1}(f(s)-f(a))ds \\
{}_tD_b^{\alpha} f(t) &= \frac{1}{\Gamma(m-\alpha)}(-\frac{d}{dt})^m \int_t^b (t-s)^{m-\alpha-1}(f(s)-f(b))ds,
\end{aligned}
$$

are called the left respectively the right Liouville Riemann fractional derivatives of function f, where $m \in \mathbb{N}^*$ with $m-1 \leq \alpha < m$ and Γ is Euler Gamma function.

In general, the operators ${}_aD_t^\alpha$, ${}_tD_b^\alpha$ do not satisfy semigroupal properties with respect to the concatenation operation. Thus, we define the derivative operators on the function spaces where the semigroupal properties hold.

The functions:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}(\frac{d}{dt})^m \int_{-\infty}^t (t-s)^{m-\alpha-1}(x(s)-x(0))ds, \quad 0 \in (-\infty, t)$$
$${}^*D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)}(-\frac{d}{dt})^m \int_t^\infty (s-t)^{m-\alpha-1}(x(s)-x(0))ds, \quad 0 \in (t, \infty)$$

are called the left, respectively the right fractional derivative of α order for function f.

If $\overline{supp}(f) = C(a,b)$, then $D_t^\alpha f(t) =_a D_t^\alpha f(t)$, ${}^*D_t^\alpha f(t) =_t D_b^\alpha f(t)$.

We define the seminorms:

$$|x|_{J_L^\alpha(R)} = ||D_t^\alpha||_{L^2(R)}, \quad |x|_{J_R^\alpha(R)} = ||^*D_t^\alpha||_{L^2(R)}$$

and the norms:

$$||x||_{J_L^\alpha(R)} = (||x||^2_{L^2(R)} + |x|^2_{J_L^\alpha(R)}), \quad ||x||_{J_R^\alpha(R)} = (||x||^2_{L^2(R)} + |x|^2_{J_L^\alpha(R)}),$$

where $J_L^\alpha(R)$, respectively $J_R^\alpha(R)$ denotes the closure of $C_0^\infty(R)$ with respect to $||\cdot||_{J_L^\alpha(R)}$, respectively $||\cdot||_{J_R^\alpha(R)}$.

From the above definitions we have the following [2]:

Proposition 2.1. *1. Let $I \subset R$ and $J_{L,0}^\alpha(I)$, $J_{R,0}^\alpha(I)$ be the closure of $C_0^\infty(I)$ in accordance with the respective norms. Then, for any $f \in J_{L,0}^\beta(I)$, $0 < \alpha < \beta$, respectively for any $f \in J_{R,0}^\beta(I)$, $0 < \alpha < \beta$, the relation*

$$D_t^\beta f(t) = D_t^\alpha D_t^{\beta-\alpha} f(t),$$

respectively, the relation

$${}^*D_t^\beta f(t) =^* D_t^{\alpha *} D_t^{\beta-\alpha} f(t)$$

holds;

2. If $\lim\limits_{n\to\infty} \alpha_n = p \in \mathbb{N}^*$, *then*

$$\lim_{n\to\infty}(D_t^{\alpha_n} f(t)) = D_t^p f(t), \quad \lim_{n\to\infty}(^*D_t^{\alpha_n} f(t)) =^* D^p f(t);$$

3. (i) If $f(t) = c$, $t \in [a,b]$, *then* $D_t^\alpha f(t) = 0$;

(ii) If $f_1(t) = t^\gamma$, $t \in [a,b]$, *then* $D_t^\alpha f_1(t) = \dfrac{t^{\gamma-\alpha}\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}$;

4. If f_1, f_2 *are analytical functions on* $[a,b]$, *then:*

$$D_t^\alpha (f_1 f_2)(t) = \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array}\right) D_t^{\alpha-k} f_1(t) \left(\frac{d}{dt}\right)^k f_2(t),$$

where $\left(\dfrac{d}{dt}\right)^k = \dfrac{d}{dt} \circ \ldots \circ \dfrac{d}{dt}$;

5.

$$\int_a^b f_1(t) D_t^\alpha f_2(t) dt = -\int_a^b f_2(t)^* D_t^\alpha f_1(t) dt;$$

6. If $f : [a,b] \to \mathbb{R}$ *is analytical and* $0 \in (a,b)$ *then*

$$f(t) = \sum_{h=0}^{\infty} E_\alpha(t^h) D_t^{\alpha h} f(t)|_{t=0},$$

where E_α *is the Mittag-Leffler function,* $E_\alpha(t^h) = \sum\limits_{h=0}^{\infty} \dfrac{t^{\alpha h}}{\Gamma(1+\alpha h)}$.

2.2 The fractional osculator bundle

Let $\alpha \in (0,1)$ and M a n-dimensional differentiable manifold. The parameterized curves on M, $c_1, c_2 : I \to M$, with $0 \in I$, $c_1(0) = c_2(0) \in M$ have a fractional contact α in x_0 if the relation

$$D_t^\alpha (f \circ c_1)|_{t=0} = D_t^\alpha (f \circ c_2)|_{t=o} \tag{1}$$

holds, for all $f \in \mathcal{F}(U)$ and $x_0 \in U$, where U is a local chart on M.

Preceding equality (1) defines a relation of equivalence. The classes $[c]_{x_0}^\alpha$ are called the fractional osculator space in x_0, which will be denoted by $Osc_{x_0}^{(\alpha)}(M) = T_{x_0}^\alpha(M)$.

Let $T^\alpha(M) = \bigcup\limits_{x_0 \in M} T^\alpha_{x_0}(M)$ and $\pi^\alpha : T^\alpha(M) \to M$, given by $\pi^\alpha[c]^\alpha_{x_0} = x_0$. There is a differential structure on $T^\alpha(M)$ and $(T^\alpha(M), \pi^\alpha, M)$ is a differentiable bundle space.

If U is a local chart on M with $x_0 \in U$ and $c : I \to M$ is a curve given by $x^i = x^i(t)$, $i = 1..n$, $t \in I$, then $[c]^\alpha_{x_0}$ is characterized by:

$$x^i(t) = x^i(0) + \frac{t^\alpha}{\Gamma(1+\alpha)} D^\alpha_t x^i|_{t=0}, \quad t \in (-\varepsilon, \varepsilon).$$

The coordinates of $[c]^\alpha_{x_0}$ on $(\pi^\alpha)^{-1}(U) \subset T^\alpha_{x_0}(M)$ are $(x^i, y^{i(\alpha)})$, where

$$x^i = x^i(0), y^{i(\alpha)} = \frac{1}{\Gamma(1+\alpha)} D^\alpha_t x^i(t)|_{t=0}, \quad i = 1..n.$$

From Proposition 2.1 and and the definition of $T^\alpha(M)$ we have:

Proposition 2.2. *1. If $0 < \alpha < \beta$ then $T^{(\alpha)}(M) \subset T^{(\beta)}(M)$;*

2. If $\lim\limits_{n\to\infty} \alpha_n = 1$ then $\lim\limits_{n\to\infty} T^{(\alpha_n)}(M) = T(M)$.

Let (x^i), $i = 1..n$ be the coordinate functions on U and $(dx^i)_{i=1..n}$ be the base of 1-forms $\mathcal{D}^1(U)$ and $\left(\frac{\partial}{\partial x^i}\right)_{i=1..n}$ the base of the vector fields $\mathcal{X}(U)$. For $f : U \to \mathbb{R}$ and $\alpha \in (0,1)$, the fractional derivative with respect to x^i is defined by:

$$D^\alpha_{x^i} f(x) = \\ = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x^i} \int_{a^i}^{x^i} \frac{f(x^1,\dots,x^{i-1},s,x^{i+1},\dots,x^n) - f(x^1,\dots,x^{i-1},a^i,x^{i+1},\dots,x^n)}{(x^i - s)^\alpha} ds \tag{2}$$

where $U_{ab} = \{x \in U, a^i \leq x^i \leq b^i, i = 1..n\} \subset U$.

From (2) we have:

Proposition 2.3. *1. If $f^i_1 = (x^i)^\gamma$ then $(D^\alpha_{x^i} f^i_1)(x) = \frac{(x^i)^{j-\alpha}\Gamma(1+\alpha)}{\Gamma(1+\gamma-\alpha)}$;*

2. If $f^j_2 = \frac{(x^j)^\alpha}{\Gamma(1+\alpha)}$ then $(D^\alpha_{x^i} f^j)(x) = \delta^j_i$;

3. $D^\alpha_{x^i}(D^\alpha_{x^j} f)(x) = D^\alpha_{x^j}(D^\alpha_{x^i} f)(x)$, $i, j = 1..n$.

We consider the functions $(x^i)^\alpha \in \mathcal{F}(U)$ and $d(x^i)^\alpha = \alpha(x^i)^{\alpha-1}dx^i \in \mathcal{D}^1(U)$, $i = 1..n$. The fractional exterior derivative is the operator $d^\alpha : \mathcal{F}(U) \to \mathcal{D}^1(U)$ given by [1]:

$$d^\alpha f = d(x^i)^\alpha D^\alpha_{x^i}(f).$$

Let $D^\alpha_{x^i} : \mathcal{D}^1(U) \to \mathcal{D}^1(U)$ be the operator given by:

$$D^\alpha_{x^i}(a_j dx^j) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} D^{\alpha-k}_{x^i}(a_j) \left(\frac{\partial}{\partial x^i}\right)^k (dx^j). \tag{3}$$

From (3) and

$$\left(\frac{\partial}{\partial x^i}\right)^k (dx^j) = d(\left(\frac{\partial}{\partial x^i}\right)^k (x^j)) = 0, k \geq 1$$

we obtain:

$$D^\alpha_{x^i}(a_j dx^j) = D^\alpha_{x^i}(a_j)dx^j. \tag{4}$$

Let $d^\alpha : \mathcal{D}^1(U) \to \mathcal{D}^2(U)$ be the operator given by:

$$d^\alpha(a_j dx^j) = d(x^i)^\alpha \wedge D^\alpha_{x^i}(a_j dx^j). \tag{5}$$

From (4) and (5) we can deduce:

$$\begin{aligned} &d^\alpha(a_j dx^j) = D^\alpha_{x^i}(a_j)d(x^i)^\alpha \wedge dx^j \\ &d^\alpha(b_j d(x^j)^\alpha) = D^\alpha_{x^i}(b_j)d(x^i)^\alpha \wedge d(x^j)^\alpha. \end{aligned} \tag{6}$$

Proposition 2.4. *Let U, $\overline{U}$, $U \cap \overline{U} \neq \varnothing$ be two charts on M, $x \in U \cap \overline{U}$ and the change of local chart given by:*

$$\overline{x}^i = \overline{x}^i(x^1, ..., x^n), \ rang\left(\frac{\partial \overline{x}^i}{\partial x^j}\right) = n. \tag{7}$$

With respect to (7) the following relations:

$$\begin{aligned} &d(x^i)^\alpha = \overset{\alpha}{J}{}^i_j(x,\overline{x})d(\overline{x}^j)^\alpha \\ &D^\alpha_{x^i} = \overset{\alpha}{J}{}^j_i(\overline{x},x)D^\alpha_{\overline{x}^j} \\ &\overset{\alpha}{J}{}^i_j(x,\overline{x}) \cdot \overset{\alpha}{J}{}^j_k(\overline{x},x) = \delta^i_k \\ &\overset{\alpha}{J}{}^i_j(x,\overline{x}) = (x^i)^{\alpha-1}\frac{\partial x^i}{\partial \overline{x}^j}\frac{1}{(\overline{x}^j)^{1-\alpha}}, \end{aligned}$$

hold, where

$$\overset{\alpha}{J}{}^i_j(x,\overline{x}) = \frac{1}{\Gamma(1+\alpha)} D^{\alpha}_{\overline{x}^j}(x^i)^{\alpha}.$$

Let $\mathcal{X}^{\alpha}(U)$ be the module of the fractional vector fields generated by the operators $\{D^{\alpha}_{x^i}\}_{i=1..n}$. A fractional field of vectors $\overset{\alpha}{X} \in \mathcal{X}^{\alpha}(U)$ has the form $\overset{\alpha}{X} = \overset{\alpha}{X}{}^i D^{\alpha}_{x^i}$, where $\overset{\alpha}{X}{}^i \in \mathcal{F}(U), i = 1..n$. Under a change of local chart it changes by $\overline{\overset{\alpha}{X}}{}^i = \overset{\alpha}{J}{}^i_j(x,\overline{x}) X^j$.

The fractional differential equation associated to the fractional field of vectors $\overset{\alpha}{X}$ is:

$$D^{\alpha}_t x^i(t) = \overset{\alpha}{X}{}^i(x(t)),\ i = 1..n. \tag{8}$$

The fractional differential equation (8) with initial conditions has solutions [2].

3 The fractional osculator bundle of higher order. Geometrical structures.

3.1 The fractional k-osculator bundle, $k \geq 1$.

The parameterized curves on M, $c_1, c_2 : I \to M$, with $0 \in I$, $c_1(0) = c_2(0) = x_0 \in M$ have a fractional contact of k order in x_0 if for any $f \in \mathcal{F}(U)$, the following relations:

$$D^{\alpha a}_t (f \circ c_1)|_{t=0} = D^{\alpha a}_t (f \circ c_2)|_{t=0},\ a = 1..k$$

hold, where $x_0 \in U$ and U is a local chart on M.

The classes $([c]^{\alpha a}_{x_0})_{a=1..k}$ are called the fractional osculator space of k order and they will be denoted by $Osc^{\alpha k}_{x_0}(M) = \overset{\alpha k}{E}_{x_0}$.

We consider $\overset{\alpha k}{E} = \bigcup_{x_0 \in M} \overset{\alpha k}{E}_{x_0}$ and $\pi^{\alpha k}_0 : \overset{\alpha k}{E} \to M$ given by $\pi^{\alpha k}_0([c]^{\alpha a}_{x_0})_{a=1..k} = x_0$. There is a differentiable structure on $\overset{\alpha k}{E}$ and $(\overset{\alpha k}{E}, \pi^{\alpha k}_0, M)$ is a differentiable bundle.

If U is a local chart on M with $x_0 \in U$ and $c : I \to M$ is a curve given by $x^i = x^i(t)$, $i = 1..n$, $t \in I$, then a class $([c]^{\alpha a}_{x_0})_{a=1..k}$ is given by the curve:

$$x^i(t) = x^i(0) + \sum_{a=1}^{k} \frac{t^{\alpha a}}{\Gamma(1+\alpha k)} D^{\alpha a}_t x^i(t)|_{t=0},\ t \in (-\varepsilon, \varepsilon).$$

In $(\pi^{\alpha k}_0)^{-1}(U) \subset \overset{\alpha k}{E}$, the coordinates of $([c]^{\alpha a}_{x_0})_{a=1..k}$ are $(x^i, y^{i(\alpha a)})$, $i = 1..n$, $a = 1..k$, where

$$x^i = x^i(0),\ y^{i(\alpha a)} = \frac{1}{\Gamma(1+\alpha k)} D^{\alpha a}_t x^i(t),\ i = 1..n, a = 1..k.$$

Proposition 3.1. *Let U, $\overline{U}$, $U \bigcap \overline{U} \neq \emptyset$ be two charts on M and*

$$\overline{x}^i = \overline{x}^i(x^1, ..., x^n),\ i = 1..n, det\left(\frac{\partial \overline{x}^i}{\partial x^j}\right) \neq 0 \tag{9}$$

the coordinates transformation. The coordinates transformation on $(\pi^{\alpha k}_0)^{-1}$ $(U \bigcap \overline{U})$ are given by:

$$\begin{aligned}
&\overline{y}^{i(\alpha)} = \overset{\alpha}{J}{}^i_j(x, \overline{x}) y^{j(\alpha)} \\
&\frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} \overline{y}^{i(\alpha a)} = \Gamma(1+\alpha) \overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, x) y^{j(\alpha)} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, y^{\alpha}) y^{j(2\alpha)} + \\
&+ ... + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, y^{\alpha b}) y^{j(\alpha b)} + ... + \frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} y^{i(\alpha a)}, \\
&a = 2..k,\ b = 2..k,\ b \leq a,
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&\overset{\alpha}{J}{}^i_j(x, \overline{x}) = D^{\alpha}_{\overline{x}^j}(x^i) \\
&\overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, y^{\alpha(b-1)}) = D^{\alpha}_{y^{j(\alpha(b-1))}} \overline{y}^{i(\alpha(a-1))},\ a, b = 2...k, b \leq a, \\
&\overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, x) = D^{\alpha}_{x^j} \overline{y}^{i(\alpha(a-1))},\ i, j = 1...n.
\end{aligned}$$

From the definition of fractional osculator bundle we can deduce that if $\underset{n\to\infty}{lim}\, \alpha_n = 1$ then $\underset{n\to\infty}{lim} \overset{\alpha_n}{E} = E = Osc^k(M)$. The bundle space (E, π, M) was defined and studied in [6].

3.2 Geometrical structures on $\overset{\alpha k}{E}$.

Let $\pi^{\alpha k}_{\alpha h} : \overset{\alpha k}{E} \to \overset{\alpha h}{E}$, $h < k$, be the projections given by:

$$\pi^{\alpha k}_{\alpha h}(x, y^{(\alpha)}, ..., y^{(\alpha k)}) = (x, y^{(\alpha)}, ..., y^{(\alpha h)})$$

and the operator $d^\alpha \pi^{\alpha k}_{\alpha h} : \mathfrak{X}^\alpha(\overset{\alpha k}{E}) \to \mathfrak{X}^\alpha(\overset{\alpha h}{E})$:

$$d^\alpha \pi^{\alpha k}_{\alpha h} = \Gamma(1+\alpha)(d(x^i)^\alpha D^\alpha_{x^i} + \sum_{a=1}^{h} d(y^{i(\alpha a)})^\alpha D^\alpha_{y^{i(\alpha a)}}), h < k,$$

where $\mathfrak{X}^\alpha(\overset{\alpha k}{E})$ is the module of fractional vector fields on $\overset{\alpha k}{E}$.

We consider $\mathcal{V}^{\alpha k}_{\alpha h} = Kerd^\alpha \pi^{\alpha k}_{\alpha h}$, $h = 0, 1, ..., k-1$ and its base given by $\{D^\alpha_{y^{i(\alpha(k+1))}}, ..., D^\alpha_{y^{i(\alpha k)}}\}$, $i = 1..n$. From the definition of $\mathcal{V}^{\alpha k}_{\alpha h}$ we get:

$$\mathcal{V}^{\alpha k}_{\alpha(k-1)} \subset \mathcal{V}^{\alpha k}_{\alpha(k-2)} \subset ... \subset \mathcal{V}^{\alpha k}_{\alpha} \subset \mathcal{V}^{\alpha k}_{0}$$
$$d^\alpha \pi^{\alpha k}_{\alpha h}(D^\alpha_{x^i}) = D^\alpha_{x^i},\ d^\alpha \pi^{\alpha k}_{\alpha h}(D^\alpha_{y^{i(b)}}) = D^\alpha_{y^{i(b)}},\ b = 1..h.$$

From Proposition 3.1 we obtain:

Proposition 3.2. *Under the change of local chart (9), the operators $D^\alpha_{x^i}$, $D^\alpha_{y^{i(\alpha a)}}$, $i = 1..n$, $a = 1..n$, change by:*

$$\begin{aligned} D^\alpha_{x^i} &= \sum_{a=1}^{k} \overset{\alpha}{J}{}^j_i(\overline{y}^{(\alpha a)}, x) D^\alpha_{\overline{y}^{j(\alpha a)}} + \overset{\alpha}{J}{}^j_i(\overline{x}, x) D^\alpha_{\overline{x}^j} \\ D^\alpha_{y^{i(\alpha a)}} &= \sum_{b=1}^{k} \overset{\alpha}{J}{}^j_i(\overline{y}^{\alpha b}, y^{\alpha a}) D^\alpha_{\overline{y}^{j(\alpha b)}}, \quad a = 1..k. \end{aligned} \tag{11}$$

From Proposition 3.2 we can deduce that $\mathcal{V}^{\alpha k}_{\alpha h}$ has geometrical character.

The following fractional fields of vectors:

$$\begin{aligned} &\overset{\alpha}{\Gamma} = y^{i(\alpha)} D^\alpha_{y^{i(\alpha k)}} \\ &\overset{2\alpha}{\Gamma} = \Gamma(1+\alpha) y^{i(\alpha)} D^\alpha_{y^{i(\alpha(k-1))}} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} y^{i(2\alpha)} D^\alpha_{y^{i(\alpha k)}} \\ &\dots \\ &\overset{\alpha k}{\Gamma} = \Gamma(1+\alpha) y^{i(\alpha)} D^\alpha_{y^{i(\alpha)}} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} y^{i(2\alpha)} D^\alpha_{y^{i(2\alpha)}} + \dots + \frac{\Gamma(\alpha(k-1))}{\Gamma(\alpha)} y^{i(\alpha k)} D^\alpha_{y^{i(\alpha k)}} \end{aligned} \tag{12}$$

are called Liouville fractional fields of vectors.

From (10) and (11) the fields $\overset{\alpha a}{\Gamma}$, $a = 1..k$ have geometrical character and $\overset{\alpha}{\Gamma} \in \mathcal{V}_0^{\alpha k}$, $\overset{2\alpha}{\Gamma} \in \mathcal{V}_\alpha^{\alpha k}$, ..., $\overset{\alpha k}{\Gamma} \in \mathcal{V}_{\alpha(k-1)}^{\alpha k}$.

The operators $\overset{\alpha k}{J} : \mathcal{X}^\alpha(\overset{\alpha k}{E}) \to \mathcal{X}^\alpha(\overset{\alpha k}{E})$ with the properties:

$$\overset{\alpha k}{J}(D^\alpha_{x^i}) = D^\alpha_{y^{i(\alpha)}},\ \overset{\alpha k}{J}(D^\alpha_{y^{i(\alpha a)}}) = D^\alpha_{y^{i(\alpha(a+1))}},\ a = 1..k-1,\ \overset{\alpha k}{J}(D^\alpha_{y^{i(\alpha k)}}) = 0 \quad (13)$$

is called αk fractional tangent structure.

From (11), (12) and (13) we have:

Proposition 3.3. *αk-fractional tangent structure has the following properties:*

1. $\overset{\alpha k}{J}$ *has a geometrical character;*
2. *rang*$(\overset{\alpha k}{J}) = kn$, $\quad \overset{\alpha k}{J} \circ ... \circ \overset{\alpha k}{J} = 0$;
3. $\overset{\alpha k}{J}(\overset{\alpha k}{\Gamma}) = \overset{\alpha(k-1)}{\Gamma}$, ..., $\overset{\alpha k}{J}(\overset{2\alpha}{\Gamma}) = \overset{\alpha}{\Gamma}$, $\overset{\alpha k}{J}(\overset{\alpha}{\Gamma}) = 0$.

The fractional field of vectors $\overset{\alpha k}{S} \in \mathcal{X}^\alpha(\overset{\alpha k}{E})$ is called αk-fractional spray if $\overset{\alpha k}{J}(\overset{\alpha k}{S}) = \overset{\alpha k}{\Gamma}$. From (12) and (13) we obtain the form of $\overset{\alpha k}{S}$:

$$\begin{aligned}\overset{\alpha k}{S} &= \Gamma(1+\alpha) y^{i(\alpha)} D^\alpha_{x^i} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} y^{i(2\alpha)} D^\alpha_{y^{i(\alpha)}} + ... + \\ &+ \frac{\Gamma(\alpha(k-1))}{\Gamma(\alpha)} y^{i(\alpha k)} D^\alpha_{y^{i(\alpha(k-1))}} - \frac{\Gamma(\alpha k)}{\Gamma(\alpha)} G^i(x, y^{(\alpha)}, ..., y^{(\alpha k)}) D^\alpha_{y^{i(\alpha k)}}.\end{aligned} \quad (14)$$

Proposition 3.4. *The αk-fractional spray uniquely defines the fractional differential equation given by:*

$$\frac{1}{\Gamma(1+\alpha k)} D_t^{\alpha(k+1)} x^i(t) + G^i(x, D_t^\alpha x, ..., \Gamma(1+\alpha(k-1)) D_t^{\alpha k} x) = 0.$$

Let $\overset{\alpha k}{\mathcal{N}}$ be the submodule of $\mathcal{X}^\alpha(\overset{\alpha k}{E})$ so that:

$$\mathcal{X}^\alpha(\overset{\alpha k}{E})|_{(\pi_0^{\alpha k})^{-1}(U)} = \mathcal{N}^{\alpha k} \oplus \mathcal{V}_0^{\alpha k}|_{(\pi_0^{\alpha k})^{-1}(U)}.$$

The submodule $\overset{\alpha k}{\mathcal{N}}$ is called fractional nonlinear connection.

We consider $l^{\alpha k}: \mathcal{X}^{(\alpha)}(U) \to \mathcal{X}^{\alpha}(\overset{\alpha k}{E})|_{(\pi_0^{\alpha k})^{-1}(U)}$, $l^{\alpha k}(D^{\alpha}_{x^i}) = \Delta^{\alpha k}_{x^i}$, $i = 1..n$, where

$$\Delta^{\alpha k}_{x^i} = D^{\alpha}_{x^i} - \sum_{a=1}^{k} \overset{\alpha a}{N}{}^j_i D_{y^{i(\alpha a)}},\ i = 1..n. \tag{15}$$

From (11) and (15) we obtain:

$$\Delta^{\alpha k}_{\overline{x}^i} = \overset{\alpha}{J}{}^j_i(x, \overline{x}) \Delta^{\alpha k}_{x^i}.$$

The functions $(\overset{\alpha a}{N}{}^j_i)_{i,j=1..n;a=1..k}$ are called the coefficients of fractional nonlinear connection.

Let $\overset{\alpha k}{\mathcal{N}}$ be the vertical submodule given by:

$$\overset{\alpha k}{\mathcal{N}}_0 = \overset{\alpha k}{\mathcal{N}},\ \overset{\alpha k}{\mathcal{N}}_a = \overset{\alpha k}{J}(\mathcal{N}^{\alpha k}_{a-1}),\ a = 1..k-1, \tag{16}$$

where $\overset{\alpha k}{\mathcal{N}}$ is the submodule defined by fractional nonlinear connection.

From (16) we have:

$$\mathcal{X}^{\alpha}(\overset{\alpha k}{E})|_{(\pi_0^{\alpha k})^{-1}(U)} = \overset{\alpha k}{\mathcal{N}}_0 \oplus ... \oplus \overset{\alpha k}{\mathcal{N}}_{k-1} \oplus \mathcal{V}^{\alpha k}_0|_{(\pi_0^{\alpha k})^{-1}(U)},\ U \subset M. \tag{17}$$

In what follows we will use a base adapted to the decomposition (17). Then:

$$\Delta^{\alpha k}_{y^{i(k)}} = \overset{\alpha k}{J}(\Delta^{\alpha k}_{x^i}),\ \Delta^{\alpha k}_{y^{i(\alpha a)}} = \overset{\alpha k}{J}(\Delta^{\alpha k}_{y^{i(a-1)}}),\ a = 2..k,\ D^{\alpha}_{y^{(\alpha)}},\ i = 1..n,$$

with

$$\Delta^{\alpha k}_{x^i} \in \overset{\alpha k}{\mathcal{N}}_0,\ \Delta^{\alpha k}_{y^{i(\alpha a)}} \in \overset{\alpha k}{\mathcal{N}}_a,\ a = 2..k-1,\ D^{\alpha}_{y^{i(\alpha)}} \in \mathcal{V}^{\alpha k}_0,\ i = 1..n$$

and

$$\begin{aligned}
&\overset{\alpha k}{\Delta}_{y^{i(\alpha)}} = D^{\alpha}_{y^{i(\alpha)}} - \overset{\alpha}{N}{}^j_i D^{\alpha}_{y^{i(2\alpha)}} - ... - \overset{\alpha(k-1)}{N}{}^j_i D^{\alpha}_{y^{j(\alpha k)}} \\
&\overset{\alpha k}{\Delta}_{y^{i(2\alpha)}} = D^{\alpha}_{y^{i(2\alpha)}} - ... - \overset{\alpha(k-1)}{N}{}^j_i D^{\alpha}_{y^{j(\alpha k)}} \\
&\cdots \\
&\overset{\alpha k}{\Delta}_{y^{i(\alpha(k-1))}} = D^{\alpha}_{y^{i(\alpha(k-1))}} - \overset{\alpha}{N}{}^j_i D^{\alpha}_{y^{j(\alpha k)}}.
\end{aligned} \tag{18}$$

The dual base of (18) is:

$$\begin{aligned}
&\overset{\alpha k}{\delta} y^{i(\alpha)} = d(y^{i(\alpha)})^{\alpha} + \overset{\alpha}{M}{}^{i}_{j} d(x^{j})^{\alpha}\\
&\overset{\alpha k}{\delta} y^{i(2\alpha)} = d(y^{i(2\alpha)})^{\alpha} + \overset{\alpha}{M}{}^{i}_{j} d(y^{i(\alpha)})^{\alpha} + \overset{2\alpha}{M}{}^{i}_{j} d(x^{j})^{\alpha}\\
&\dots\\
&\overset{\alpha k}{\delta} y^{i(\alpha k)} = d(y^{i(\alpha k)})^{\alpha} + \overset{\alpha}{M}{}^{i}_{j} d(y^{i(\alpha(k-1))})^{\alpha} + \cdots + \overset{\alpha k}{M}{}^{i}_{j} d(x^{j})^{\alpha},
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
&\overset{\alpha}{M}{}^{i}_{j} = \overset{\alpha}{N}{}^{i}_{j},\ \overset{2\alpha}{M}{}^{i}_{j} = \overset{2\alpha}{N}{}^{i}_{j} + \overset{\alpha}{N}{}^{i}_{k} \overset{\alpha}{N}{}^{h}_{j}\\
&\dots\\
&\overset{\alpha k}{M}{}^{i}_{j} = \overset{\alpha k}{N}{}^{i}_{j} + \overset{\alpha(k-1)}{N_{h}} \overset{\alpha}{N}{}^{h}_{j} + \cdots + \overset{\alpha}{N}{}^{i}_{h} \overset{\alpha(k-1)}{N}{}^{h}_{j}.
\end{aligned}$$

The functions $\overset{\alpha a}{M}{}^{i}_{j}$, $a = 1..k$ are called dual coefficients of fractional nonlinear connection.

From (14) and (19) we obtain:

Proposition 3.5. *A αk-fractional spray , $\overset{\alpha k}{S}$, with the components $G^{i}(x, y^{(\alpha)}, ..., y^{(\alpha k)})$, determines a fractional nonlinear connection with the dual coefficients given by:*

$$\begin{aligned}
&\overset{\alpha}{M}{}^{i}_{j} = D^{\alpha}_{y^{i(\alpha)}} G^{i}\\
&\overset{2\alpha}{M}{}^{i}_{j} = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} (\overset{\alpha k}{S}(\overset{\alpha}{M}{}^{i}_{j}) + \overset{\alpha}{M}{}^{i}_{l} \overset{\alpha}{M}{}^{l}_{j})\\
&\dots\\
&\overset{\alpha k}{M}{}^{i}_{j} = \frac{\Gamma(\alpha(k-1))}{\Gamma(\alpha k)} (\overset{\alpha k}{S}(\overset{\alpha(k-1)}{M}{}^{i}_{j}) + \overset{\alpha}{M}{}^{i}_{l} \overset{\alpha(k-1)}{M}{}^{l}_{j}).
\end{aligned}$$

We consider the adapted base given by (18) and the operator $\overset{\alpha k}{\mathcal{L}}$ defined by:

$$\begin{aligned}
&\overset{\alpha k}{\mathcal{L}}_{\Delta^{\alpha b}_{x^{i}}} (\overset{\alpha k}{\Delta}_{y^{j(\alpha a)}}) = \overset{(\alpha k)}{L}{}^{h}_{ji} \overset{\alpha k}{\Delta}_{y^{h(\alpha a)}}\\
&\overset{\alpha k}{\mathcal{L}}_{\Delta^{\alpha k}_{y^{i(\alpha b)}}} (\overset{\alpha k}{\Delta}_{y^{j(\alpha a)}}) = \overset{(\alpha b)}{C}{}^{h}_{ji} \overset{\alpha k}{\Delta}_{y^{h(\alpha a)}},\ \alpha = 0, 1, ..., k,\ b = 1..k,
\end{aligned} \tag{20}$$

where $y^{i(0)} = x^i$. The coefficients $(\overset{(\alpha k)}{L}{}^h_{ji}, \overset{(\alpha b)}{C}{}^h_{ji})$ are called the fractional coefficients of linear connection N.

A distinguished fractional tensor field of type $(0, k)$ is given by the following expression:

$$\overset{\alpha k}{g} = \overset{\alpha k}{g}{}_{i_0 i_1 \dots i_k} d(y^{i_0(0)})^\alpha \otimes \overset{\alpha k}{\delta} y^{i_1(\alpha)} \otimes \dots \otimes \overset{\alpha k}{\delta} y^{i_k(\alpha k)},$$

where $\overset{\alpha k}{\delta} y^{i(\alpha a)}$, $a = 0..k$ are given by (19) and $y^{i(0)} = x^i$.

The covariant derivative with respect to fractional nonlinear connection N of $\overset{\alpha k}{g}$ is defined by:

$$g^{\alpha k}_{i_0 i_1 \dots i_k | m} = \overset{\alpha k}{\Delta}_{y^{m(0)}}(g^{\alpha k}_{i_0 i_1 \dots i_k}) - \overset{(\alpha k)}{L}{}^j_{i_0 m} \overset{\alpha k}{g}{}_{j i_1 \dots i_k},$$
$$g^{\alpha k}_{i_0 i_1 \dots i_k \overset{(\alpha b)}{|} m} = \overset{\alpha k}{\Delta}_{y^{m(\alpha b)}}(g^{\alpha k}_{i_0 i_1 \dots i_k}) - \overset{\alpha k}{C}{}^h_{i_1 m} g^{\alpha k}_{i_0 h \dots i_k} - \dots - \overset{\alpha k}{C}{}^h_{i_k m} g^{\alpha k}_{i_0 \dots h}.$$

A fractional metric structure on $\overset{\alpha k}{E}$ is a fractional field of tensors of type $(0, 2)$, $\overset{\alpha k}{g} = \overset{\alpha k}{g}{}_{ij} d(x^i)^\alpha \otimes d(x^j)^\alpha$, with $\overset{\alpha k}{g}{}_{ij}(x, y^{(\alpha)}, ..., y^{(\alpha k)})$, which is symmetric and positively defined.

The fractional Sasaki lift of $\overset{\alpha k}{g}$ is the fractional field of tensors given by:

$$\overset{\alpha k}{G} = \overset{\alpha k}{g}{}_{ij} d(x^i)^\alpha \otimes d(x^j)^\alpha + \sum_{a=1}^{k} \overset{\alpha k}{g}{}_{ij} \delta y^{i(\alpha)} \otimes \delta y^{j(\alpha)}.$$

If:

$$\overset{\alpha k}{g}{}_{ij|m} = 0, \; \overset{\alpha k}{g}{}_{ij \overset{\alpha k}{|} m} = 0$$

hold, then the fractional linear connection (20) is called metrical.

Proposition 3.6. *On $\overset{\alpha k}{E}$ there is a unique metrical fractional linear connection N with respect to metrical structure $\overset{\alpha k}{G}$ with the property:*

$$\overset{(\alpha k)}{L}{}^i_{jl} = \overset{(\alpha k)}{L}{}^i_{lj}, \; \overset{(\alpha a)}{C}{}^i_{jl} = \overset{(\alpha a)}{C}{}^i_{lj}, \; a = 1..k.$$

The coefficients $\overset{(\alpha k)}{L}{}^{i}_{jl}$, $\overset{(\alpha k)}{C}{}^{i}_{jl}$ *have the expressions:*

$$\overset{(\alpha k)}{L}{}^{i}_{jl} = \frac{1}{2}\overset{\alpha k}{g}{}^{is}(\overset{\alpha k}{\Delta}_{x^j}\overset{\alpha k}{g}_{sl} + \overset{\alpha k}{\Delta}_{x^l}\overset{\alpha k}{g}_{js} - \overset{\alpha k}{\Delta}_{x^s}\overset{\alpha k}{g}_{jl})$$

$$\overset{(\alpha a)}{C}{}^{i}_{jl} = \frac{1}{2}\overset{\alpha k}{g}{}^{is}(\overset{\alpha k}{\Delta}_{y^{j(\alpha a)}}\overset{\alpha k}{g}_{sl} + \overset{\alpha k}{\Delta}_{y^{l(\alpha a)}}\overset{\alpha k}{g}_{js} - \overset{\alpha k}{\Delta}_{y^{s(\alpha a)}}\overset{\alpha k}{g}_{jl}),\ a = 1..k.$$

4 Lagrange space $\overset{\alpha k}{L}$. Applications.

4.1 The fractional Euler-Lagrange equation.

A fractional Lagrangian of k order, $k \in \mathbb{N}^*$, on the differentiable manifold M, is a differentiable map $L : \overset{\alpha k}{E} \to \mathbb{R}$ on $\overset{\tilde{\alpha k}}{E} = \{(x, y^{(\alpha)}, ..., y^{(\alpha a)}) \in \overset{\alpha k}{E}, rang||y^{i(\alpha)}|| = 1\}$. Also, L is continuous in the points of $\overset{\alpha k}{E}$ for which $y^{i(\alpha)}$ is zero. Then,

$$g_{ij}(x, y^{(\alpha)}, ..., y^{(\alpha k)}) = \frac{1}{2}D^{\alpha}_{y^{i(\alpha)}}D^{\alpha}_{y^{j(\alpha)}}L$$

is d-fractional field of tensors on $\overset{\alpha k}{E}$. The Lagrangian L is regular if $rang(g_{ij}) = n$ on $\overset{\tilde{\alpha k}}{E}$.

Let $c : t \in [0,1] \to (x^i(t)) \in M$ be a parameterized curve so that $Imc \subset U$, where U is a chart on M. The extension of curve c to $\overset{\alpha k}{E}$, $\overset{\alpha k}{c}$, is the following differentiable map:

$$\overset{\alpha k}{c} : t \in [0,1] \to (x^i(t), y^{i(\alpha)}(t), ..., y^{i(\alpha k)}(t)) \in \overset{\tilde{\alpha k}}{E}. \tag{21}$$

The action of L along the curve $\overset{\alpha k}{c}$ is given by:

$$I(\overset{\alpha k}{c}) = \int_0^1 L(x(t), y^{(\alpha)}(t), ..., y^{(\alpha k)}(t))dt.$$

Let $c_\varepsilon : t \in [0,1] \to (x^i(t,\varepsilon)) \in M$ be the family of curves so that $Imc_\varepsilon \subset U$ and $c_\varepsilon(0) = c(0)$, $y^{i(\alpha a)}(0) = y^{i(\alpha a)}(1) = 0$, $a = 1..k-1$, where ε is a sufficiently small number in absolute value. The action on $\overset{\alpha k}{c_\varepsilon}$ is:

$$I(\overset{\alpha k}{c_\varepsilon}) = \int_0^1 L(x_\varepsilon(t), y^{(\alpha)}_\varepsilon(t), ..., y^{(\alpha k)}_\varepsilon(t))dt.$$

A necessary condition for $I(\overset{\alpha k}{c})$ to be an extreme fractional value for $I(\overset{\alpha k}{c_\varepsilon})$ is:

$$D^{\alpha}_{\varepsilon} I(\overset{\alpha k}{c_\varepsilon})|_{\varepsilon} = 0.$$

By direct calculus we obtain:

Proposition 4.1. *The curve $c : t \in [0,1] \to (x^i(t)) \in M$ has the property that the action $I(\overset{\alpha k}{c})$ is an extreme value of $I(\overset{\alpha k}{c_\varepsilon})$ if $(x^i(t))$, $i = 1..n$ is a solution of fractional Euler-Lagrange equation:*

$$D^{\alpha}_{x^i} L + \sum_{a=1}^{k} (-1)^a d^{\alpha a}_t (D^{\alpha}_{y^{i(\alpha a)}} L) = 0, \quad i = 1..n \tag{22}$$

where $d^{\alpha a}_t = \sum\limits_{b=1}^{a} y^{i(\alpha b)} D^{\alpha}_{y^{i(\alpha(b-1))}}$, $y^{i(0)} = x^i$, $a = 1..h$.

A necessary condition for $I(\overset{\alpha k}{c})$ to be an extreme value for $I(\overset{\alpha k}{c_\varepsilon})$ is:

$$\frac{dI(\overset{\alpha k}{c_\varepsilon})}{d\varepsilon}|_{\varepsilon=0} = 0.$$

Proposition 4.2. *The curve $c : t \in [0,1] \to (x^i(t)) \in M$ has the property that the action $I(\overset{\alpha k}{c})$ is an extreme value of $I(\overset{\alpha k}{c_\varepsilon})$ if $(x^i(t))$, $i = 1..n$ is a solution of fractional Euler-Lagrange equation:*

$$\frac{\partial L}{\partial x^i} + \sum_{a=1}^{k} (-1)^a d^{\alpha}_t \left(\frac{\partial L}{\partial y^{i(\alpha a)}}\right) = 0,\ i = 1..n, \tag{23}$$

where $d^{\alpha}_t = \sum\limits_{a=1}^{h} y^{i(\alpha a)} D^{\alpha}_{y^{i(\alpha(a-1))}}$, $y^{i(0)} = x^i$.

Example. We consider the fractional differential equation:

$$\frac{c\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha} + a_1\Gamma(1+2\alpha) y^{2\alpha} + a_2\Gamma(1+3\alpha) y^{3\alpha} + a_3\Gamma(1+4\alpha) y^{4\alpha} = 0. \tag{24}$$

Equation (24) is the fractional Euler-Lagrange equation (22) for the fractional Lagrange function:

$$L = \frac{c}{1+\gamma-\alpha} x^{\gamma} - a_1\Gamma(1+2\alpha)(y^{\alpha})^{\alpha} + a_2\Gamma(1+3\alpha)(y^{2\alpha})^{\alpha} - a_3\Gamma(1+4\alpha)(y^{3\alpha})^{\alpha}.$$

Equation (24) is the fractional Euler Lagrange equation (23) for the fractional Lagrange function:

$$L = \frac{c\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)(\gamma-\alpha+1)} x^{\gamma-\alpha-1} - \frac{a_1}{2}\Gamma(1+2\alpha)(y^{\alpha})^2 + \frac{a_2}{2}\Gamma(1+3\alpha)(y^{2\alpha})^2 - $$
$$- \frac{a_3}{2}\Gamma(1+4\alpha)(y^{3\alpha})^2.$$

Along the curve c we define the operators:

$$\overset{0}{E}_i = D^{\alpha}_{x^i} + \sum_{a=1}^{h}(-1)^a \frac{1}{\Gamma(1+\alpha a)} d_t^{\alpha a}(D^{\alpha}_{y^{i(\alpha a)}}), \quad i = 1..n$$
$$\overset{\alpha}{E}_i = \sum_{a=1}^{h}(-1)^a \frac{1}{\Gamma(1+\alpha a)} d_t^{\alpha a}(D^{\alpha}_{y^{i(\alpha a)}}), \quad i = 1..n$$
$$\ldots$$
$$\overset{\alpha h}{E}_i = (-1)^k \frac{1}{\Gamma(1+\alpha k)} d_t^{\alpha k}(D^{\alpha}_{y^{i(\alpha k)}}), \quad i = 1..n$$

which have the property:

Proposition 4.3. *The operators $\overset{0}{E}_i(L)$, ...,$\overset{\alpha k}{E}_i(L)$, $i = 1..n$, are d-fractional fields of covectors for any differentiable Lagrangian of order αk, $\overset{\alpha k}{L}$, along the extension $\overset{\alpha k}{c}$ of curve c.*

d-fields of covectors $\overset{\alpha}{E}_i(L)$, ...,$\overset{\alpha k}{E}_i(L)$ are called Craig and Synge d-fractional fields of covectors.

Proposition 4.4. *1. d-fractional field of covectors, $\overset{\alpha(k-1)}{E_i}(L)$, has the form:*

$$\overset{\alpha(k-1)}{E_i}(L) = (-1)^{k-1}\frac{1}{\Gamma(1+\alpha(k-1))}(D^{\alpha}_{y^{i(\alpha(k-1))}}L - \overset{\alpha}{\Gamma}(D^{\alpha}_{y^{i(\alpha(k))}}L) - g_{ij}y^{j(\alpha(k+1))}),$$

$i = 1..n$, where $\overset{\alpha}{\Gamma}$ is given (12).

2. The system of fractional differential equations:

$$g^{ij}\overset{\alpha(k-1)}{E}_j(L) = 0,\ i = 1..n$$

determines a αk fractional spray $\overset{\alpha k}{S}$ on the curve $\overset{\alpha k}{c}$, given by (21):

$$\overset{\alpha k i}{G} = \frac{\Gamma(\alpha)}{\Gamma(1+\alpha k)\Gamma(1+\alpha)} g^{ij}[\overset{\alpha}{\Gamma}(D^{\alpha}_{y^{j(\alpha k)}}L) - D^{\alpha}_{y^{j(\alpha(k-1))}}],\ i,j = 1..n.$$

4.2 The prolongation of Riemann, Finsler and Lagrange fractional structures to fractional bundle of k order.

The pair $\overset{\alpha}{\mathcal{R}} = (M, \overset{\alpha}{g})$ is called Riemann fractional structure, where M is a differentiable manifold of n dimension and $\overset{\alpha}{g} = (\overset{\alpha}{g}_{ij})$ is a fractional field of tensors, which means that under a change of local chart on M, the system of functions $\overset{\alpha}{g}_{ij}$ change by:

$$\overset{\alpha}{g}_{ij}(\overline{x}) = \overset{\alpha l}{J}_i(\overline{x}, x)\overset{\alpha h}{J}_j(\overline{x}, x)\overset{\alpha}{g}_{lh}(x)$$

and $\overset{\alpha}{g}_{ij} = \overset{\alpha}{g}_{ji}$ with $(\overset{\alpha}{g}_{ij})$ is positively defined. The fractional Christofel symbols $\overset{\alpha}{\gamma}{}^{l}{}_{ij}$ of $\overset{\alpha}{g}$ are:

$$\overset{\alpha}{\gamma}{}^{l}_{ij} = \frac{1}{2}\overset{\alpha}{g}{}^{ls}(D^{\alpha}_{x^i}\overset{\alpha}{g}_{sj} + D^{\alpha}_{x^j}\overset{\alpha}{g}_{is} - D^{\alpha}_{x^s}\overset{\alpha}{g}_{ij}).$$

The prolongation of $\overset{\alpha}{g}$ to $\overset{\alpha k}{E}$ is the fractional field of tensors $\overset{\alpha k}{g}$ with the property:

$$(\overset{\alpha k}{g} \circ \pi^{\alpha k}_0)(x, y^{(\alpha)}, ..., y^{(\alpha k)}) = \overset{\alpha}{g}(x),\ \forall (x, y^{(\alpha)}, ..., y^{(\alpha k)}) \in (\pi^{\alpha k}_0)^{-1}(U).$$

Proposition 4.5. *There are fractional nonlinear connections $\overset{\alpha k}{N}$ on $\overset{\alpha k}{E}$ which are determined only by $\overset{\alpha}{g}$. One of them is:*

$$\begin{aligned}
&\overset{\alpha}{M}{}^i_j = \overset{\alpha}{\gamma}{}^i_{jm} y^{(\alpha)m},\\
&\overset{2\alpha}{M}{}^i_j = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\overset{\alpha}{\Gamma}(\overset{\alpha}{M}{}^i_j) + \overset{\alpha}{M}{}^i_h \overset{\alpha}{M}{}^h_j),\\
&\dots\\
&\overset{\alpha k}{M}{}^i_j = \frac{\Gamma(\alpha(k-1))}{\Gamma(\alpha k)}(\overset{\alpha}{\Gamma}(\overset{\alpha(k-1)}{M}{}^i_j) + \overset{\alpha}{M}{}^i_h \overset{\alpha(k-1)}{M}{}^h_j).
\end{aligned} \tag{25}$$

For $k = 1$ the coefficients of fractional nonlinear connection $\overset{\alpha}{N}$ on $\overset{\alpha}{E}$ are $\overset{\alpha}{M}{}^i_j = \gamma^i_{jh} y^{h(\alpha)}$ and for $k = 2$ the coefficients are:

$$\overset{\alpha}{M}{}^i_j = \gamma^i_{jh} y^{h(\alpha)}, \overset{2\alpha}{M}{}^i_j = \gamma^i_{jh} y^{h(2\alpha)} + \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(D^{\alpha}_{x^h}\gamma^i_{jp} + \gamma^i_{hl}\gamma^l_{jp}) y^{h(\alpha)} y^{p(2\alpha)}.$$

The pair $\overset{\alpha}{\mathfrak{F}} = (M, \overset{\alpha}{F})$ is called Finsler fractional structure, where M is a differentiable manifold of dimension n, and $\overset{\alpha}{F} : \overset{\alpha}{E} \to M$ is called fundamental fractional function. We consider the prolongation of $\overset{\alpha}{F}$, $\overset{\alpha k}{F} : \overset{\alpha k}{E} \to \mathbb{R}$ given by:

$$(\overset{\alpha k}{F} \circ \pi^{\alpha k}_{\alpha})(x, y^{(\alpha)}, ..., y^{(\alpha k)}) = \overset{\alpha}{F}(x, y^{(\alpha)})$$

and the prolongation of fundamental fractional tensor:

$$\overset{\alpha}{\gamma}_{ij}(x, y^{(\alpha)}) = \frac{1}{2} D^{\alpha}_{y^{i(\alpha)}}(D^{\alpha}_{y^{j(\alpha)}} \overset{\alpha}{F}{}^2)$$

given by:

$$(\overset{\alpha k}{\gamma}_{ij} \circ \pi^{\alpha k}_{0})(x, y^{(\alpha)}, ..., y^{(\alpha k)}) = \overset{\alpha}{\gamma}_{ij}(x, y^{\alpha}).$$

Let $\overset{\alpha}{\gamma}{}^h_{ij}(x, y^{(\alpha)})$ be the Christoffel symbols of $(\overset{\alpha}{\gamma}_{ij})$, given by:

$$\overset{\alpha}{\gamma}{}^h_{ij}(x, y^{(\alpha)}) = \frac{1}{2} \overset{\alpha}{\gamma}{}^{ls}(D^{\alpha}_{x^i} \overset{\alpha}{\gamma}_{ij} + D^{\alpha}_{x^j} \overset{\alpha}{\gamma}_{is} - D^{\alpha}_{x^s} \overset{\alpha}{\gamma}_{ij}),$$

where $(\overset{\alpha}{\gamma}{}^{ls}) = (\overset{\alpha}{\gamma}_{ls})^{-1}$.

The coefficients of nonlinear fractional connection (fractional Cartan coefficients) are:

$$\overset{\alpha}{G}{}^i_j = \frac{1}{2} D^{\alpha}_{y^{j(\alpha)}}(\overset{\alpha}{\gamma}{}^i_{pm} y^{p(\alpha)} y^{m(\alpha)}).$$

Proposition 4.6. *There is a nonlinear fractional connection on* $\overset{\tilde{\alpha k}}{E} = \overset{\alpha k}{E} \diagdown \{0\} = \{(x, y^{(a)}, ..., y^{(\alpha a)}) \in \overset{\alpha k}{E}, rang||y^{i(\alpha)}|| = 1\}$ *which only depends on the fundamental fractional function* $\overset{\alpha}{F}$ *of the fractional Finsler space. One of them has the dual coefficients given by:*

$$\overset{\alpha}{M}{}^i_j = \overset{\alpha}{G}{}^i_j,$$
$$\overset{2\alpha}{M}{}^i_j = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\overset{\alpha}{\Gamma}(\overset{\alpha}{M}{}^i_j) + \overset{\alpha}{G}{}^i_m \overset{\alpha}{M}{}^m_j),$$
$$\dots$$
$$\overset{\alpha k}{M}{}^i_j = \frac{\Gamma(\alpha(k-1))}{\Gamma(\alpha k)}(\overset{\alpha}{\Gamma}(\overset{\alpha(k-1)}{M}{}^i_j) + \overset{\alpha}{G}{}^i_m \overset{\alpha(k-1)}{M}{}^m_j).$$

The pair $\overset{\alpha}{\mathcal{L}} = (M, \overset{\alpha}{L})$ is called Lagrange fractional structure, where $\overset{\alpha}{L} : \overset{\alpha}{E} \to \mathbb{R}$.

The prolongation of $\overset{\alpha}{L}$ to $\overset{\alpha k}{E}$ is defined by:

$$(\overset{\alpha k}{L} \circ \pi^{\alpha k}_{\alpha})(x, y^{(\alpha)}, ..., y^{(\alpha k)}) = \overset{\alpha}{L}(x, y^{(\alpha)})$$

and the prolongation of fundamental tensor:

$$\overset{\alpha}{g}_{ij}(x, y^{(\alpha)}) = \frac{1}{2} D^{\alpha}_{y^{i(\alpha)}} D^{\alpha}_{y^{j(\alpha)}} \overset{\alpha}{L}(x, y^{(\alpha)})$$

to $\overset{\alpha k}{E}$ is:

$$(\overset{\alpha k}{g}_{ij} \circ \pi^{\alpha k}_{\alpha})(x, y^{(\alpha)}, ..., y^{(\alpha k)}) = \overset{\alpha}{g}_{ij}(x, y^{(\alpha)}).$$

Considering the integral action $I(\overset{\alpha}{c}) = \int_0^1 L(x(t), y^{(\alpha)}(t))dt$ on a parameterized curve c, the fractional Euler-Lagrange equations (22) are:

$$y^{i(2\alpha)} = \overset{\alpha}{G^i}(x, y^{(\alpha)}),\ i = 1..n$$

where

$$\overset{\alpha}{G^i}(x, y^{(\alpha)}) = g^{im}(D^{\alpha}_{y^{m(\alpha)}} D^{\alpha}_{x^j} y^{j(\alpha)} - D^{\alpha}_{x^m} L),\ i = 1..n$$

$(g^{im}) = (g_{im})^{-1}$, $g_{im} = D^{\alpha}_{y^{i(\alpha)}} D^{\alpha}_{y^{m(\alpha)}} L$.

The functions:

$$\overset{\alpha}{G^i_j}(x, y^{(\alpha)}) = D^{\alpha}_{y^{j(\alpha)}} G^i(x, y^{(\alpha)}),\ i, j = 1..n \tag{26}$$

are the first dual coefficients of fractional nonlinear connection $\overset{\alpha}{\mathcal{N}}$ on $\overset{\tilde{\alpha}}{E}$ which only depends on the fundamental function $\overset{\alpha}{L}$ of the Lagrange space $\overset{\alpha}{\mathcal{L}}$.

We obtain the result:

Proposition 4.7. *There are nonlinear fractional connections on $\overset{\tilde{\alpha k}}{E}$ which only depend on the fundamental function $\overset{\alpha}{L}$ of the Lagrange space $\overset{\alpha}{\mathcal{L}}$. One of them has the dual coefficients (25), where $\overset{\alpha}{G^i_j}$ are given by (26).*

5 Conclusions.

The study conducted in this paper takes into account the geometrical character of the introduced objects. In the case $M = \mathbb{R}$, using the methods from this paper, the information concerning fractional differential systems which describe concrete processes was obtained in [3]. The results from the present paper will permit the study of other geometrical objects which are described with the help of the fractional derivative.

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