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ON THE DISCRETE-CONTINUOUS DYNAMICAL SYSTEMS

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1. Introduction. In the last twenty years the study of the dynamical systems using difference equations enjoyed of a special interest.Starting from a suggestion of T.D.Lee[5] one can say that the aspects of finitude of the phisical reality impose rather the discrete phenomena than the continuous ones.Among these the time-discrete evolution is distinguished.Some physical models (harmonic oscillator, Newtonian potential system) but especially the economical models (Samuelson's Bussiness Cycle, Hicks's trade cycle, etc) and the biological and statistical models (waiting process, birth and death processes, neuronal networks, etc) require a specific approach, a discrete description according to the essence of the phenomenon.

In order to obtain such a description for the discrete phenomena there are two ways: the first one uses methods and schemas of discretization of the continuous description for an evolution system, evidently with the preservation of their special structures; the second one is to go from the discrete model to the difference equations.The latter directly application imposes a specific variational calculus and proper arguments.

The utilization of equations of the discrete mechanics in the study of some classes of Hamiltonian systems was done by Greenspan(1973-1974) [2] and Labudde(1980) [4].Then Lee(1987) utilized the Lagrangian formalism in order to develop a discrete mechanics in which the trajectories of the system are supposed continuous and piecewise linear (the energy is constant on each segment of trajectory in each step of time).

Y.Wu(1990) [6] showed that the symplectic integration schemas for the Hamiltonian systems admit a natural discrete variational principle.Shibberu (1993) [5] describes a discrete-time theory for the Hamiltonnian dynamical systems based on a version of the variational principle ("stationary action" principle). He deduced the Hamilton equations in discrete-time.In [5] the time is considered in the action integral as a dependent variable.On the other hand it is assured the conservation of the energy in the medium values of the trajectory by considering so-called midpoint scheme utilized in the discretization and the integration of some differential equations.For the Newton potential systems of Lee, D'Innocenzo and others [3] Shibberu finds again identical continuous piecewise linear trajectories for the coordinate-position q, but the Shibberu's procedure determines also continuous trajectories for the coordinate-momentum p.

The main idea of our study is a direct treatise of the dynamical systems of discrete-continuous type our aim is to: formulate a discrete-continuous variational principle and deduce the discrete-continuous Euler-Lagrange equations (solution for the direct problem); establish the self-adjointing conditions of Helmholtz type for difference equation systems (solution for the inverse problem); study the question of the conservation laws for discrete Lagrangians and formulate a discrete version of the Noether's theorem; obtain the discrete Euler-Lagrange-Hamilton equations based on a result established in the section 2.

The text contains several examples including the discrete case for the mentioned references.

2. Discrete-Continuous Euler Equations. Let $\{\tau_k\}_{k\in[0,N]}$, $[0,N]=\{0,1,2,...,N\}$, be a division of the interval $[\tau_0,\tau_N]\subset \mathbf{R}$, where

$$\tau_k=\tau_0+k\Delta\tau\ ,\ \Delta\tau=\frac{\tau_N-\tau_0}{N} \tag{2.1}$$

and let $\mathcal{R}=[\tau_0,\tau_N]\times[a,b]\subset\mathbf{R}^2$ be the two-dimensional network whose an arbitrary element $(\tau_k,s)\in\mathcal{R}$ is denoted by (k,s), k$\in[0,N]$, s$\in[a,b]$. For a function $y:\mathcal{R}\longrightarrow\mathbf{R}^n, C^1$–differentiable with respect to $s\in[a,b]$, we denote:

$$y(k,s)=(y^i(k,s)),\overline{y}(k,s)=\frac{1}{2}(y(k+1,s)+y(k,s))$$

$$y^1(k,s)=\frac{1}{\Delta\tau}\cdot(y(k+1,s)-y(k,s)) \tag{2.2}$$

$$\dot{y}(k,s)=\frac{dy(k,s)}{ds}\ ,\ \dot{\overline{y}}(k,s)=\frac{d\overline{y}(k,s)}{ds}.$$

The set

$$\mathcal{L}^2(\mathcal{R})=\{y:\mathcal{R}\longrightarrow\mathbf{R}^n|\int_a^b(\sum_{k=0}^N\delta_{ij}\cdot y^j(k,s)\Delta\tau)ds<\infty\} \tag{2.3}$$

endowed with the scalar product

$$(y,z)=\int_a^b(\sum_{k=0}^{N}\delta_{ij}\cdot y^i(k,s)\cdot z^j(k,s)\cdot\Delta\tau)ds$$

is a Hilbert space.The tangent space of the manifold

$$\Omega=\{y\in\mathcal{L}^2(\mathcal{R})\ ,\ y(k,a)=y_1(k)\ ,\ y(k,b)=y_2(k),k\in[1,N-1], \tag{2.4}$$

$$y(0,s)=y_3(s)\ ,\ y(N,s)=y_4(s)\ ,\ s\in[a,b]\}$$

in a point $y\in\Omega$ is

$$T_y\Omega=\{\eta:\mathcal{R}\longrightarrow\mathbf{R}^n\ ,\ \eta(k,a)=0\ ,\ \eta(k,b)=0\ ,\ k\in[1,N-1], \tag{2.5}$$

$$\eta(0,s)=0\ ,\ \eta(N,s)=0\ ,\ s\in[a,b]\}$$

For a function $F:\Omega\longrightarrow\mathbf{R}$, the variation of F is the function $\delta F:T_y\longrightarrow\mathbf{R}$, given by:

$$\delta F(\eta)=\frac{dF(y_{(\varepsilon)})}{d\varepsilon}|_{\varepsilon=0};\eta(k)=\frac{dy_{(\varepsilon)}(k)}{d\varepsilon}|_{\varepsilon=0}, \tag{2.6}$$

where $y_{(\varepsilon)}\in\Omega$ with $\varepsilon\in I\subset\mathbf{R},0\in I,y_{(0)}=y$.The point $y\in\Omega$ is critical(or stationary) for F if at this point $\delta F=0$.

For $\eta\in T_y\Omega$, we denote

$$\overline{\eta}(k,s)=\frac{1}{2}(\eta(k+1,s)+\eta(k,s)),$$

$$\eta^1(k,s)=\frac{1}{\Delta\tau}(\eta(k+1,s)-\eta(k,s)), \tag{2.7}$$

$$\overset{\circ}{\overline{\eta}}=\frac{d\overline{\eta}(k,s)}{ds},\overset{\circ}{\eta}(k,s)=\frac{d\eta(k,s)}{ds},(k,s)\in R$$

Let $\overline{\Omega}=\{\overline{y}(k,s),y\in\Omega\ ,\ (k,s)\in\mathcal{R}\}$, $\Omega^1=\{y^1(k,s)\ ,\ y\in\Omega\ ,\ (k,s)\in\mathcal{R}\}$, $\overset{\circ}{\overline{\Omega}}=\{\overset{\circ}{\overline{y}}(k,s)\ ,\ y\in\Omega\ ,\ (k,s)\in\mathcal{R}\}$ and $L:\mathcal{R}\times\Omega\times\overline{\Omega}\times\Omega^1\times\overset{\circ}{\overline{\Omega}}\longrightarrow\mathbf{R}$, be a function of class C^2 with respect to the variables from $\Omega, \overline{\Omega}, \Omega^1, \overset{\circ}{\Omega}$ and $s \in [a,b]$. By denoting

$$L(k,s) = L(k,s,y(k,s)\ ,\ \overline{y}(k,s), y^1(k,s), \overset{\circ}{y}(k,s))\ ,\ (k,s) \in \mathcal{R}\ , \tag{2.9}$$

the functional

$$\mathcal{A}(y) = \int_a^b (\sum_{k=0}^{N-1} L(k,s) \cdot \Delta\tau) ds \tag{2.10}$$

is called the **action of L with respect to** $y \in \Omega$.

Theorem 2.1.(discrete-continuous variation principle). *The function* $y \in \Omega$ *is a critical point for* $\mathcal{A}(y)$ *iff*

$$\frac{d(L(k,s) + L(k-1,s))}{dy^i(k,s)} - \frac{d}{ds}(\frac{\partial(L(k,s) + L(k-1,s))}{\partial \overset{\circ}{y}{}^i(k,s)}) = 0, \tag{2.11}$$

$$(k,s) \in \mathcal{R}\ ,\ i = 1, \ldots, n.$$

Proof: Let $y_{(\varepsilon)} \in \Omega$, with $\varepsilon \in I \subset \mathbf{R}, 0 \in I, y_{(0)} = y$, and $\eta(k,s) = \frac{dy_{(\varepsilon)}(k,s)}{d\varepsilon}|_{\varepsilon=0}$. The variations of the function $\mathcal{A}(y)$ is

$$\delta\mathcal{A}(y)(\eta) = \int_a^b (\sum_{k=1}^{N-1} (\frac{d(L(k,s) + L(k-1,s))}{dy^i(k,s)} - \tag{2.12}$$

$$-\frac{d}{ds}(\frac{\partial(L(k,s) + L(k-1,s))}{\partial \overset{\circ}{y}{}^i(k,s)}) \cdot \eta^i(k)\Delta\tau) ds$$

By (2.12), we obtain (2.11).

For $L(k,s) = L(s, y(s), \dot{y}(s))$, $s \in [a,b]$ we obtain from (2.11), the (continuous) Euler-Lagrange equations. For $L(k,s) = L(k, y(k), \overline{y}(k), \overline{y}^1(k)$, $k \in [0, N-1]$, we obtain from (2.11) the discrete Euler equations:

$$E_i(k) = \frac{d(L(k) + L(k-1))}{dy^i(k)} = 0\ ,\ i = 1, \ldots, n. \tag{2.13}$$

For $\Delta\tau = 1$, $\tau_0 = 1$, $\tau_N = N$, from (2.13), we deduce the discrete Euler equations given in [1]:

$$\frac{\partial L(k)}{\partial y^i(k)} - \frac{\partial L(k)}{\partial y^{1i}(k)} + \frac{\partial L(k-1)}{\partial y^{1i}(k-1)} = 0 \tag{2.14}$$

Examples : 1. For $L(k) = \frac{1}{2} \cdot (-1)^k \cdot y^1(k)^2 + \frac{1}{2} y(-1)^k \cdot y(k)^2$, $k \in [0, N-1]$, $y(k) \in \mathbf{R}$, from (2.14) one obtains:

$$y^1(k) = y(k-1),$$

which together with the initial condition, $y(0) = 1, y(1) = 1$, represents the low of the Fibonacci recurrence:
2. For $L(k) = \frac{1}{2} \cdot \rho^{k+2} \cdot y^1(k)^2$, $\rho = \frac{\lambda}{\mu}$, $\lambda > 0$, $\mu > 0$, $y(k) \in \mathbf{R}$, (2.14) implies

$$\lambda \cdot y(k+1) - (\lambda + \mu) \cdot y(k) + \mu \cdot y(k-1) = 0$$

which represents the equation of a waiting process in the case of statistic equilibrium.
3. For the discretization of the Lagrangian of a Newtonian potential system, given in [7], $L(k) = \frac{1}{2} \cdot m \frac{q^1(k^2)}{t^1(k)} - V(\overline{q}(k)) \cdot t^1(k), (q,t) : \mathcal{R} \longrightarrow \mathbf{R}^2$, by using (2.13) it results

$$\begin{aligned} & m \cdot \frac{1}{\Delta\tau} \cdot (\overline{v}(k) - \overline{v}(k-1)) = -\frac{1}{2}\left(\frac{\partial V(\overline{q}(k))}{\partial \overline{q}(k)} \cdot t^1(k) + \frac{\partial V(\overline{q}(k-1))}{\partial \overline{q}(k-1)}\right) \\ & \frac{1}{2} \cdot m \cdot \overline{v}(k)^2 + V(\overline{q}(k)) = \frac{1}{2} \cdot m \cdot \overline{v}(k-1)^2 + V(\overline{q}(k-1)), \end{aligned} \tag{2.15}$$

where $\overline{v}(k) = \frac{q^1(k)}{t^1(k)}$.The difference equations (2.15) represent the discret equations of the mechanics,given in [2],[3],[4],[5].

3. Helmholtz conditions for systems of difference equations. Let $F_i : \mathcal{R} \times \Omega \times \overline{\Omega} \times \Omega^1 \longrightarrow \mathbf{R}, i = 1, \ldots, n$, be $\mathbf{C}^2$ function with respect to the variables from $\Omega, \overline{\Omega}, \Omega^1$, given by:

$$F_i(k) = F_i(k, y(k-1), y(k), \overline{y}(k-1), \overline{y}(k), y^1(k-1), y^1(k)) , \tag{3.16}$$

$$i = 1, \ldots, n.$$

From (2.2) and (3.16) it results

$$\frac{dF_i(k)}{dy^j(k-1)} = \frac{\partial F_i(k)}{\partial y^j(k-1)} + \frac{1}{2} \cdot \frac{\partial F_i(k)}{\partial \overline{y}^j(k-1)} - \frac{1}{\Delta\tau} \cdot \frac{\partial F_i(k)}{\partial y^{1j}(k-1)} =$$

$$= a_{ij}(k, k-1)$$

$$\frac{dF_i(k)}{dy^j(k)} = \frac{\partial F_i(k)}{\partial y^j(k)} + \frac{1}{2}\cdot\frac{\partial F_i(k)}{\partial \overline{y}^j(k)} + \frac{1}{2}\cdot\frac{\partial F_i(k)}{\partial \overline{y}^j(k)} + \frac{1}{\Delta\tau}\cdot\frac{\partial F_i(k)}{\partial y^{1j}(k-1)} - \tag{3.17}$$

$$-\frac{1}{\Delta\tau}\cdot\frac{\partial F_i(k)}{\partial y^{1j}(k)} = b_{ij}(k,k)$$

$$\frac{dF_i(k)}{dy^j(k+1)} = \frac{1}{2}\cdot\frac{\partial F_i(k)}{\partial \overline{y}^j(k)} + \frac{1}{\Delta\tau}\cdot\frac{\partial F_i(k)}{\partial y^{1j}(k)} = C_{ij}(k,k+1)$$

We call variation forms associated to (3.16) the function system

$$\delta F_i(k)(\eta) = \frac{dF_i(k,\varepsilon)}{d\varepsilon}|_{\varepsilon=0}\ , \quad i = 1,\dots,n. \tag{3.18}$$

where $F_i(k,\varepsilon) = F_i(k, y_{(\varepsilon)}(k-1), y_{(\epsilon)}(k), \overline{y}_{(\epsilon)}(k-1), \overline{y}_{(\epsilon)}(k), \overline{y}^1(k-1), \overline{y}^1(k))$, and $y_{(\varepsilon)} \in \Omega$,is a one-parametric family of elements of Ω with $\epsilon \in I \subset \mathcal{R}, 0 \in I$,such that $y_{(0)} = y$ and $\eta(k) = \frac{dy_{(\varepsilon)}(k)}{d\varepsilon}|_{(\epsilon=0)}$
From (3.18) we have:

$$\delta F_i(k)(\eta) = a_{ij}(k,k-1)\cdot\eta^j(k-1) + b_{ij}(k,k)\cdot\eta^j(k) \tag{3.19}$$

$$+c_{ij}(k,k+1)\cdot\eta^j(k+1)\ , \quad i,j = 1,\dots,n.$$

For another one-parametric family $\widetilde{y}_{(\varepsilon)}, \varepsilon \in I$, the variation forms are given by

$$\delta\widetilde{F}_i(k)(\widetilde{\eta}) = \widetilde{a}_{ij}(k,k-1)\cdot\widetilde{\eta}^j(k-1) + \widetilde{b}_{ij}(k,k)\cdot\widetilde{\eta}^j(k) \tag{3.20}$$

$$+\widetilde{c}_{ij}(k,k+1)\cdot\widetilde{\eta}^j(k+1).$$

The variation forms $\{\delta\widetilde{F}_i(k)(\widetilde{\eta})\}, i = 1,\dots,n$, are called adjoint of $\{\delta F_i(k)(\eta)\}, i = 1,\dots,n$ if there is the function $Q(\eta,\widetilde{\eta}) : \mathcal{R} \longrightarrow \mathbf{R}$ such that

$$\widetilde{\eta}^i(k)\cdot\delta F_i^{(k)}(\eta) - \eta^i(k)\cdot\widetilde{F}_i(k)(\widetilde{\eta}) = Q(\eta,\widetilde{\eta})(k+1) - Q(\eta,\widetilde{\eta})(k)\ , \ k \in \mathcal{R} \tag{3.21}$$

for any $\eta, \widetilde{\eta} \in T_y\Omega$.

Theorem 3.2. *The variation forms* $\{\delta\widetilde{F}_i(k)(\widetilde{\eta})\}_{i=1,\ldots,n}$, *are adjoint to the variation forms* $\{\delta F_i(k)(\eta)\}_{i=1,\ldots,n}$, *iff*

$$\widetilde{a}_{ij}(k,k+1) = c_{ji}(k,k+1)\ ,$$

$$\widetilde{b}_{ji}(k,k) = b_{ji}(k,k)\ , \tag{3.22}$$

$$\widetilde{c}_{ji}(k+1,k) = a_{ji}(k+1,k,k)$$

Proof: By considering $Q(\eta,\widetilde{\eta})(k) = A_{ij}(k)\cdot\widetilde{\eta}^i(k)\cdot\eta^j(k-1) + B_{ij}(k)\cdot\eta^i(k)\widetilde{\eta}^j(k-1)$ the condition (3.21) implies for any $\eta,\widetilde{\eta}$.

$\widetilde{a}_{ij}(k,k-1) = B_{ij}(k), \widetilde{b}_{ij}(k,k) = b_{ji}(k,k), \widetilde{c}_{ij}(k,k+1) = -A_{ji}(k-1)$, $B_{ij}(k+1) = c_{ji}(k,k+1), A_{ij}(k) = -a_{ij}(k,k-1)$ and (3.22) verified.

The variations forms $\{\delta F_i(k)(\eta)\}_{i=1,\ldots,n}$, are called self-adjoint if they coincide with it adjoint variation form, that is

$$\delta F_i(k)(\eta) = \delta\widetilde{F}_i(k)(\eta)\ ,\ \ i=1,\ldots,n\ , \tag{3.23}$$

for any η.From (3.23), with (3.22), and (3.17) we have:

Proposition 3.3: *The variation forms* $\{\delta F_i(\eta)\}_{i=1,\ldots,n}$ *are self adjoint forms iff*:

$$\frac{dF_i(k)}{dy^j(k-1)} = \frac{dF_j(k-1)}{dy^i(k)}$$

$$\frac{dF_i(k)}{dy^j}(k) = \frac{dF_j(k)}{dy^i(k)}, i,j=1,\ldots,n. \tag{3.24}$$

Theorem 3.4: *The function system*

$$E_i(k) = \frac{d(L(k)+L(k-1))}{dy^i(k)}\ ,\ i=1,\ldots,n\ ,\ k\in[1,N] \tag{3.25}$$

is self-adjoint, where $L(k) = L(k,y(k),\overline{y}(k),y^1(k))$.

The relation (3.24) represents the **Helmholtz conditions** for difference equation systems with the left member given by (3.16).

Theorem 3.5 : *Let* $\{F_i(k)\}_{i=1,\dots,n}$, $k \in [1,N]$, *be a function system satisfying (3.24).There exists a function* $L(k)$, $k \in [1,N]$, *such that*

$$F_i(k) = \frac{d(L(k) + L(k-1))}{dy^i(k)} \ , \ i = 1,\dots,n \ , \ k \in [1,N].$$

Proof.: From the second relation (3.24) it results that there is $L(k)$ and $G(k-1)$ such that

$$F_i(k) = \frac{dL(k)}{dy^i(k)} + \frac{dG(k-1)}{dy^i(k)}, i = 1,\dots,n, k \in [1,N]$$

Since $L(k) = L(k, y(k), \overline{y}(k), y^1(k))$ and $G(k-1) = G(k-1, y(k-1), \overline{y}(k-1), y^1(k-1))$, from the first relation (3.24) it results $G(k) = L(k) + C, C \in \mathbf{R}$.

The relations (3.24) are necessary and sufficient conditions for the function system $\{F_i(k)\}_{i=1,\dots,n}$, $k \in [1,N]$, to stem from a discrete variational principle.The functions $\{F_i(k)\}_{i=1,\dots}, n$, satisfying (3.24) are called function steming from a discrete variational principle (d.p.v.).

The function $F(k) = F(k, y(k))$, $y(k) \in bfR$, stems from a d.v.p. The associated $L(k)$ is $L(k) = \int F(k, y(k))dy(k)$. The function $F(k) = \lambda \cdot y^1(k) - \mu \cdot y^1(k-1)$, $\lambda \neq \mu$, doesn't stem from a d.v.p.The function $\widetilde{F}(k) = c(k) \cdot F(k)$, with $c(k) = (\frac{\lambda}{\mu})^k \cdot c(0)$ stems from a d.v.p.

Let $\{F_i(k)\}_{i=1,\dots,n}$, $k \in [0, N-1]$, a function system and $\widetilde{F}_i(k) = C_i^j \cdot F_j(k)$, with $det\|C_i^j(k)\| \neq 0, \forall k \in [0, N-1]$.

Proposition 3.6 a)*The function system* $\{\widetilde{F}_i(k)\}_{i=1,\dots,n}, k \in [0, N-1]$ *stems from a d.v.p., iff the follownig relations are verified:*

$$(\frac{dC_i^h(k)}{dy^j}(k) - \frac{dC_j^h(k)}{dy^i(k)}) \cdot F_h(k) + (C_i^h(k) \cdot \delta_l^j - C_j^h(k) \cdot \delta_i^l) \cdot \frac{dF_h(k)}{dy^l(k)} = 0,$$

$$(3.26) \quad \frac{dC_i^h(k+1)}{dy^j(k)} \cdot F_h(k) - \frac{dC_i^h(k)}{dy^j(k+1)} \cdot F_h(k) + C_i^h(k+1) \cdot \frac{dF_h(k+1)}{dy^j(k)} -$$

$$C_i^h(k) \cdot \frac{dF_h(k)}{dy^j(k+1)} = 0$$

b)If the function system $\{F_i(k)\}_{i=1,\dots,n}$ comes from d.v.p. then the conditions (3.26) become:

$$(\frac{dC_i^h(k)}{dy^j(k)}-\frac{dC_j^h(k)}{dy^i(k)})\cdot F_h(k)+(C_i^h\cdot\delta_j^l-C_j^h(k)\cdot\delta_i^l)\cdot\frac{dF_h(k)}{dy^l(k)}=0$$

$$\frac{dC_i^h(k+1)}{dy^j(k)}\cdot F_h(k+1)-\frac{dC_i^h(k)}{dy^j(k+1)}\cdot F_h(k)+(C_i^h(k+1)\cdot\delta_j^l\cdot\delta_h^m-$$
$$-C_i^h(k)\cdot\delta_j^m\cdot\delta_h^l)\cdot\frac{dF_m(k+1)}{dy^l(k)}=0 \tag{3.27}$$

Proof.: The curent relations in **Proposition 3.6** are consequences of (3.24)

The matrix $\|C_j^i(k)\|$ in Proposition 3.6 is called, an integrant factor.In order to determine this integrant factor it must to consider special cases with functions of the type $C_j^i(k,y(k),\overline{y}(k))$.

Examples. The Samuelson's Bussiness Cycle is given by aid of the difference equation

(S) $F(k)=y(k+1)-c\cdot(1+v)\cdot y(k)+c\cdot v\cdot y(k-1)-1=0$

where $y(0)$, $y(1)$ are prescribed and $y(k)$ represents the income at the moment k, $0<c<1,v>0$.

The trade cycle Hicks model is given by the equation

(H) $F(k)=y(k+1)-(1-s+v)\cdot y(k)+v\cdot y(k+1)-A_0\cdot(1+g)^{k+1}=0$

where $0<s<1,v>0$.

The both models are described by aid of the function

(M) $F(k)=a\cdot y(k+1)+b\cdot y(k)+c\cdot y(k-1)+g(k)$

with $a\neq c, y(k)\in\mathbf{R}$, $y(0)$, $y(1)$ are given.

Proposition 3.7 *a)$F(k)$ is a self-adjoint form iff $a=c$;*

b)If $a\neq c$, there is $C(k)$ such that $\widetilde{F}(k)=C(k)\cdot F(k)$ is self-adjoint; $C(k)=(\frac{a}{c})^k$.The function $\widetilde{L}(k)$ given by

$$\widetilde{L}(k)=-\frac{1}{2}\cdot\frac{a^{k+1}}{c^k}\cdot y^1(k)^2+\frac{1}{2}\cdot\frac{a^k}{c^k}\cdot(a+b+c)\cdot y(k)^2+\frac{a^k}{c^k}\cdot g(k)\cdot y(k)$$

represents the Lagrangian associated to $\widetilde{F}(k)$.

For the model (S), $C(k)=(\frac{1}{c\cdot v})^k$ and for the model (H) $c(k)=\frac{1}{v^k}$.

4. Conservation laws for the Lagrange function on $\mathcal{R}\times\Omega\times\overline{\Omega}\times\Omega^1$
In this section we present the notion of the conservation law for the functions $L : \mathcal{R}\times\Omega\times\overline{\Omega}\times\Omega^1\longrightarrow\mathbf{R}$, as well as a discrete version of the Noether's theorem.

Let $L(k) = L(k, y(k), \overline{y}(k), y^1(k))$ and $E_i(k) = \frac{d(L(k)+L(k-1))}{dy^i(k)}$ $i = 1,\dots,n$, $k\in\mathcal{R}$
The function

$$L^{'}(k) = L(k) + A_i\cdot y^{1i}(k)\ ,\ A_i\in\mathbf{R} \tag{4.28}$$

describes the same discrete dynamical system like $L(k)$, because $E_i^{'}(k) = E_i(k)$, $k\in\mathcal{R}$

Let $\theta : \mathcal{R}\times\Omega\longrightarrow\mathcal{R}\times\Omega$ be the differentiable transformation given by (4.28)

$$\theta(k, y(k-1)) = (k, \widetilde{y}(k, y(k-1)))\ \ k\in\mathcal{R} \tag{4.29}$$

and θ^p, canonical prolongation of θ on $\mathcal{R}\times\Omega\times\overline{\Omega}\times\Omega^1$.

θ is said to be a **symmetry transformation** of the system generated by the Lagrangian L, if θ invaries d.v.p.According to (4.28), and $L^{'}(k)$,θ is a symmetry transformation if $(L\circ\theta^p)(k) = L(k) + L^{'}(k)$.We call a **deformation** of the symmetry θ a differentiable map $\Theta : I\times\mathcal{R}\times\Omega\longrightarrow\mathcal{R}\times\Omega$, $0\in I\subset\mathbf{R}$, such that $\Theta(0, k, y(k-1)) = \theta(k, y(k-1))$, and $\Theta_{(\varepsilon)}(k, y(k-1)) = \Theta(\varepsilon, k, y(k-1))$ are symmetry transformations of $\mathcal{R}\times\Omega$ for any $\varepsilon\in I$.Consider a pseudo-group of diffeomorphisms $\{\theta_{(\varepsilon)}\}_{\varepsilon\in I}$ on $\mathcal{R}\times\Omega$, and $\{\theta^p_{(\varepsilon)}\}_\varepsilon\in I$ the canonical prolongations $\mathcal{R}\times\Omega\times\overline{\Omega}\times\Omega^1$.

Proposition 4.8. *If $\{\theta_{(\varepsilon)}\}_{\varepsilon\in I}$ is a pseudo-group of symmetries for L, then we have:*

$$\left(\frac{dL(k)}{dy^i(k)} + \frac{1}{\Delta\tau}\cdot A_i\right)\cdot\widetilde{\eta}^i(k-1) = -\left(\frac{dL(k)}{dy^i(k+1)} - \frac{1}{\Delta\tau}\cdot A_i\right)\cdot\widetilde{\eta}^i(k) \tag{4.30}$$

where

$$\widetilde{\eta}^i(k-1) = \frac{d\widetilde{y}^i_{(\varepsilon)}(k)}{d\varepsilon}\Big|_{\varepsilon=0}\ ,\ \widetilde{\eta}^i(k) = \frac{d\widetilde{y}^i_{(\varepsilon)}(k+1)}{d\varepsilon}\Big|_{\varepsilon=0}\ . \tag{4.31}$$

Theorem 4.9: (Discrete Noether's Theorem). *A symmetry pseudo-group of L determines a function system $\mathcal{L}(k)$ which all constant along an orbit of the discrete dynamical system described by L.*

Proof.: From (4.30) it results

$$(E_i(k)-\frac{dL(k-1)}{dy^i(k)}+\frac{1}{\Delta\tau}\cdot A_i)\cdot\widetilde{\eta}^i(k-1) = -(\frac{dL(k)}{dy^i(k-1)}-\frac{1}{\Delta\tau}\cdot A_i)\cdot\widetilde{\eta}^i(k)\ ,\ k\in\mathcal{R}$$

Since $E_i(k)=0$ along an orbit of the system, we conclude that

$$\mathcal{L}(k) = (\frac{dL(k)}{dy^i(k+1)} - \frac{1}{\Delta\tau}\cdot A_i)\cdot\widetilde{\eta}^i(k)\ ,\ k\in\mathcal{R} \tag{4.32}$$

is constant because $\mathcal{L}(k-1)=\mathcal{L}(k)$, $\forall k\in\mathcal{R}$. For $n=1$ we have

Proposition 4.10 *$\{\widetilde{\eta}(k)\}_{k\in\mathcal{R}}$ satisfying (4.30) are given by*

$$\widetilde{\eta}(k) = (-1)^k\cdot\widetilde{\eta}(0)\frac{\prod_{p=1}^{k}(\frac{dL(p)}{dy(p)}+\frac{1}{\Delta\tau}A)}{\prod_{p=1}^{k}(\frac{dL(p)}{dy(p+1)}-\frac{1}{\Delta\tau}A)} \tag{4.33}$$

and the associated function is

$$\mathcal{L}(k) = (-1)^k\cdot\widetilde{\eta}(0)\frac{\prod_{p=1}^{k}(\frac{dL(p)}{dy(p)}+\frac{1}{\Delta\tau}A)}{\prod_{p=1}^{k-1}(\frac{dL(p)}{dy(p+1)}-\frac{1}{\Delta\tau}A)} \tag{4.34}$$

Examples: 1. For $L(k)=\frac{1}{2}\cdot\rho^{k+2}\cdot y^1(k)^2$, and $\Delta\tau=1$ it results $\widetilde{\eta}(k)=(-1)^k\widetilde{\eta}(0)$, $\mathcal{L}(k)=(-1)^k(\rho^{k+2}y^1(k)-A)\widetilde{\eta}(0)$

2. For $L(k)=\frac{1}{2}\frac{q^1(k)^2}{t^1(k)}-V(\overline{q}(k))t^1(k)$, from Ex.4., we have $\mathcal{L}(k)=\frac{1}{2}\overline{v}(k)^2+V(\overline{q}(k))$. A variable $y^i(k)$, (i fixed) is called a **cyclic variable** if $\frac{\partial L(k)}{\partial y^i(k)}=0$, $\frac{\partial L(k)}{\partial\overline{y}^i(k)}$.If $y^i(k)$ is a cyclic variable then $\mathcal{L}_i(k)=\frac{df(k)}{dy^i(k+1)}$ has the property $\mathcal{L}_i(k)=\mathcal{L}_i(k-1)$, $\forall k\in\mathcal{R}$.

5. Discrete Lagrange-Hamilton equations. Let $\mathcal{R}$ be a network in $\mathbf{R}$, Ω given by (2.4), $\Omega^*=\{p:\mathcal{R}\longrightarrow\mathbf{R}^{*n}|\ \sum_{k=0}^{N}\delta^{ij}\cdot p_i(k)\cdot p_j(k)<\infty,\ p(0)=b_n\ ,\ p(N)=b_2\}$ and $H:\mathcal{R}\times\Omega\times\Omega^*\longrightarrow\mathbf{R}$ a function of class C^1 with respect to the variables in Ω, Ω^*; H is called a discrete Hamilton function, denote by

$$H(k) = H(k, y(k), p(k-1)) \tag{5.35}$$

The functional

$$\mathcal{B}(y,p) = \sum_{k=0}^{N-1} (p_i(k-1)y^{1i}(k-1) - H(k))\Delta\tau \tag{5.36}$$

is called the action of H with respect to (y,p). The variations of the function $B(y,p)$ is

$$\delta\mathcal{B}(y,p)(\eta,\xi) = \sum_{k=1}^{N-1} [\xi_i(k-1)y^{1i}(k-1) + p_i(k-1)\eta^{1i}(k-1) - \\ -\frac{\partial H(k)}{\partial y^i(k)} \cdot \eta^i(k) - \frac{\partial H(k)}{\partial p_i(k-1)}\xi_i(k-1)] \cdot \Delta\tau \tag{5.37}$$

Proposition 5.11. *The function $(y,p) \in \Omega \times \Omega^*$ is a critical point for $\mathcal{B}(y,p)$, iff*

$$\begin{aligned} y^{1i}(k-1) &= \tfrac{\partial H(k)}{\partial p_i(k-1)} \\ p_i^1(k-1) &= -\tfrac{\partial H(k)}{\partial y^i(k)}\ , i = 1,\dots,n,\ k \in [1, N-1] \end{aligned} \tag{5.38}$$

The relations (5.38), represent the discrete Hamilton equations. Using the transformation associated in some regularity conditions, the discrete Hamilton equations are obtained from the discrete Euler equations [1]. Let $\mathbf{R}^r \subset \mathbf{R}^n$, Ω, $\Omega_r^*, \Omega_{n-r}^1$, and $R : \mathcal{R} \times \Omega_r^* \times \Omega_{n-r}^1 \longrightarrow \mathbf{R}$, a function of class C^1, with respect to the variables in Ω, Ω_r^*, Ω_{n-r}^1; R is called the discrete Routh function. We denote

$$R(k) = R(k, y^i(k), p_a(k-1), y^{1\alpha}(k)), \quad \begin{aligned} &i = 1,\dots,n, \\ &\alpha = r+1,\dots,n, \\ &a = 1,\dots,r. \end{aligned} \tag{5.39}$$

The functional

$$\mathcal{C}(y,p) = \sum_{k=0}^{N-1} (p_a(k-1)y^{1a}(k-1) - R(k)) \cdot \Delta\tau \tag{5.40}$$

is called the action of R with respect to (y, p, y^1). The variations of the function $\mathcal{C}(y, p)$ is

$$\delta\mathcal{C}(y,p)(\eta,\xi)=\sum_{k=1}^{N-1}\Big[\xi_a(k-1)\cdot y^{1a}(k-1)+p_a(k-1)\cdot\eta^{1a}(k-1)- \tag{5.41}$$
$$-\frac{\partial R(k)}{\partial y^i(k)}\cdot\eta^i(k)-\frac{\partial R(k)}{\partial p_a(k-1)}\cdot\xi_a(k-1)-\frac{\partial R(k)}{\partial y^{1\alpha}(k)}\cdot\eta^{1\alpha}(k)\Big]\cdot\Delta\tau$$

Proposition 5.12. *The function $(y, p) \in \Omega \times \Omega_r^*$ is a critical point for $\mathcal{C}(y, p)$, iff*

$$y^{1a}(k-1)=\frac{\partial R(k)}{\partial p_a(k-1)}\ ,\ p_a^1(k-1)=-\frac{\partial R(k)}{\partial y^\alpha(k)}\ \ a=1,\dots,r \tag{5.42}$$
$$\frac{\partial R(k)}{\partial y^\alpha(k)}-\frac{1}{\Delta\tau}\Big(\frac{\partial R(k)}{\partial y^{1\alpha}(k)}-\frac{\partial R(k-1)}{\partial y^{1\alpha}(k-1)}\Big)=0\ ,\alpha=r+1,\dots,n;$$

The relations (5.42) represent the discrete Euler-Lagrange-Hamilton (Routh) equations. For $r = 0$, the relations (5.42) represent the discrete Euler equations and for $r = n$, the relations (5.42) represent the discrete Hamilton equations.

REFERENCES

1. A h l b r a n d t, C.D. – *Equivalence of Discrete Euler Equations and Discrete Hamiltonian System*, J. of Math. Anal. and Appl. 180(1993), p. 498-517.
2. G r e e n s p a n, D. – *Discrete numerical methods in physics and engineering*, Academic Press, New York, 1974.
3. D'I n n o c e n z o, A., R e u n a, L. and R o t e l l i – *Some studies in discrete mechanics*, European J. of Physics, 8(1987), p. 245-252.
4. L a b u d d e, R.A. – *Discrete Hamilton Mechanics*, International J. of General Systems, 6(1980), p. 3-12.
5. L e e, T.D. – *Difference Equations and Conservation Laws*, J. of Statistical Physics, 46(1987), nr. 5-6, p. 843-860.
6. W u, Y. – *The discrete variations approach to the Euler-Lagrange Equations*, Computers and Mathematics with applications, 20(1990), nr. 8, p. 61-75.
7. S h i b b e r u, Y. – *Discrete-Time Hamiltonian Dynamics*, Ph. D. Thesis, Univ. of Texas at Arlington, 1992.

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