Analytical Algorithms for Synthesis of Modal Controllers by the Maximum Stability Degree Criterion

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***Abstract* - The practice of synthesis the controllers demonstrates, that for the determination of the dynamic tuning parameters of the controller is more convenient to operate with the analytical expressions with a low volume of calculations that dependent on the known parameters of control object. The analytical synthesis expressions, on the one hand have the advantage of decreasing the volume of calculation of tuning parameters (compared with the synthesis methods and algorithms that include a number of steps) and, by the other hand, using of the analytical expressions is a good alternative in case of the controllers with auto-tuning and adaptive control, where the controller retuning is done in function of the parameters variation of the control object during operation of the control system.**

Based on this consideration, in this paper it is proposed the analytical algorithms of synthesis the modal controllers, in form of algebraic expressions, for control objects with arbitrary order inertia and astatism by the maximum stability degree criterion. This criterion offers to the designed control systems an aperiodic step response, high performance and better robustness. The elaborated algorithms represent simple analytical procedures with reduced volume of calculation and without any imposing conditions to the complexity of the control object. They allow also to impose or to optimize the settling time of the designed automatic control system.

Keywords: *control system, state space representation, synthesis of the modal controllers, analytical algorithms, maximum stability degree.*

I. INTRODUCTION

State space representation has become the mathematical support in the systems theory and a source for a new series of approaches and modern methods for analysis and synthesis of control systems. This fact is due to the following issues: representation in the state space using the matrix calculations that are easy to implement on the computer; permits unitary treatment of the mono-variable and multi-variable systems, continuous and discrete systems, linear and nonlinear systems; it is used for synthesis of the controllers to the high order objects etc. The state variables $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$ are those variables that determine the future behavior of the system, when the initial state of the system and the inputs are known. For state space realization of the system it is need to be satisfied the condition of controllability and observability [1, 2, 6].

The modal control method solves the problem of synthesis the controller based on the changes of the modes to achieve the optimal control of the object. The modes are eigenvalues of the state matrix and represent the roots of the characteristic equation of the closed loop system. The obvious relation of location the eigenvalues in the complex plane of roots with the dynamics of the system makes an important the task of moving the roots to the desired area [5, 9, 10, 12].

Synthesis of the modal controller is started with determination of the characteristic polynomial $\varphi_A(p)$ of the control system state matrix A, imposing the poles (eigenvalues) $[\gamma_1, \gamma_2, ..., \gamma_n]$, that determine the desired dynamics of the design system, according to which is obtained the characteristic polynomial $\varphi_c(p)$ of the system matrix in the closed loop. For determination of the feedback vector' components k (tuning parameters) is used the Ackermann relation [2, 7, 10]

$$\begin{aligned} k &= [0\ 0\ ...\ 1]U^{-1}\varphi(A) = \\ &= [0\ 0\ ...\ 1][B, AB, ..., A^{n-1}B]^{-1}\varphi(A) \end{aligned} \qquad (1,a)$$

where $\varphi(A) = A^n + q_{n-1}A^{n-1} + ... + q_1 A + q_0 I$; U - the controllability matrix.

In case of presentation of the system in canonical controllability form, calculation of the feedback vector' components is reduced to use the following expressions [6, 10]

$$k_i = q_i - \alpha_i,\ i = 0,...,(n-1), \qquad (1,b)$$

where q_i and α_i represent the coefficients of characteristic polynomials $\varphi_c(p)$ and $\varphi_A(p)$ respectively.

Thus, using the feedback by the states is possible to modify all poles of the control system and, therefore, imposing the dynamic behavior according to the desired performance by choosing the eigenvalues of the system matrix in the closed loop. The choosing of new eigenvalues is a complex problem and using of classical methods of synthesis, for example, the dominant poles method, the responses prototype method, the analytical design of controllers etc., for the control systems with high order is met difficulties that appear in case of correlation the poles of the system with the desired performance and energetic indices required the graphic design and specialized software, but the obtained optimal parameters by these methods sometimes can not satisfy the condition of stability [9, 11, 12].

In paper [3] it is proposed a new synthesis method of the modal controller by the maximum stability degree criterion (MSD), criterion that offers to the design systems the higher performance and better robustness.

The problem of synthesis the control system in the state space by the maximum stability degree is formulated in the following way [3]. It is considered a structure of mono-variable control system with representation in the state space (Fig.1) that includes the control object with known parameters

$$\begin{cases} \dot{x} = Ax + bu, \\ y = c^T x, \end{cases} \quad (2)$$

and control algorithm

$$u(t) = -k^T x(t) = -[k_0 \quad k_1 \quad \cdots \quad k_{n-1}] \cdot [x_1 \quad x_2 \quad \ldots \quad x_n]^T, \quad (3)$$

where A is the state matrix with dimension $(n \times n)$; x - the vector of the state variables, $(n \times 1)$; u – the control value; b - the vector of control values, $(n \times 1)$; c - the vector of output values, $(n \times 1)$; k - the vector of tuning parameters, $(n \times 1)$; n - the order of the system; y - the output value.

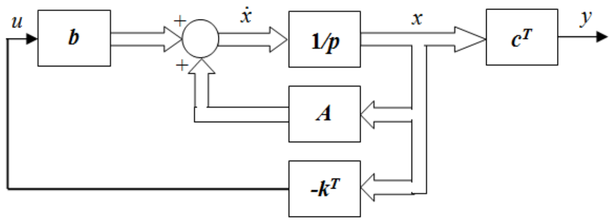

Fig. 1. The block scheme of a dynamic system in the state space.

It is necessary to determine the components of the feedback vector (tuning parameters), so as to be satisfied the condition

$$J = \max_{k_i} \eta(k_i), i = (1,\ldots,n), \quad (4)$$

where J is the maximum stability degree; η - the stability degree of the system; k_i - the components of the tuning parameters vector; n - the degree of the characteristic polynomial of the control system [4].

In conformity with method in [3], it is introduced the notion of the maximum stability degree J and using the substitution $p_i = -J \pm j\omega_k$, the desired characteristic polynomial $\varphi_c(p)$ of the design system is obtained by the decomposition of the characteristic polynomial $\varphi_A(p)$ of the A matrix in n linear factors

$$\varphi_c(p) = \prod_{k=1}^{l}(p - j\omega_k + J)(p + j\omega_k + J)\prod_{j=1}^{z}(p + J) = p^n + q_{n-1}p^{n-1} + \ldots + q_1 p + q_0, \quad (5)$$

where l is the number of conjugate pairs of complex roots; z - the number of real roots; $n = 2l + z$ - the degree of the characteristic polynomial of the design control system; $q_i = f_i(J, \omega_k), i = (0,\ldots,n-1)$.

The value of the maximum stability degree J of the designed control system is obtained from the following expression [3]

$$J = \frac{\alpha_{n-1}}{n}, \quad (6)$$

where α_{n-1} is a coefficient of the characteristic polynomial $\varphi_A(p)$.

The values of the tuning parameters k_i are determined in conformity with expressions (1, *b*).

II. Analytical Algorithms for Synthesis of the Modal Controllers

If it is imposed the problem to design of the control system in the state space by the error, the solution of this problem depends on the structure of control system, where the control object can be with inertia and astatism.

The transfer function of control object with inertia is given in the following form:

$$H_F(s) = \frac{k}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}, \quad (7)$$

where k is the transfer coefficient; $a_0, a_1, \ldots, a_n$ - the coefficients of the transfer function of control object, n - the order of control object. For the control object with inertia and astatism we have the coefficient $a_n = 0$.

The standard controllable form of representation in the state space of the object (7), normalized by the a_0, is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \quad (8)$$

$$y = [\beta_0 \quad 0 \quad 0 \quad \ldots \quad 0]x = \beta_0 x_1.$$

where $\alpha_0 = \frac{a_n}{a_0}; \alpha_1 = \frac{a_{n-1}}{a_0}; \ldots; \alpha_{n-1} = \frac{a_1}{a_0}; \beta_0 = \frac{k}{a_0}$.

For the model object with inertia and astatism in expression (8) we have $\alpha_0 = 0$.

For the control object with inertia and astatism the structural block scheme of the control system in the state space is represented in the Figure 2 [2, 6]. To amplify the error signal, in direct connection is included the proportional block k_0. The control algorithm is determined by the following expression

$$u(t) = -[k_1 \quad k_2 \quad \cdots \quad k_{n-1}] \cdot [x_2 \quad x_3 \quad \ldots \quad x_n]^T + k_0 \varepsilon. \quad (9)$$

If the reference of the control system with object with inertia is the step signal, then to obtain the stationary error null it is necessary to add in the controller structure

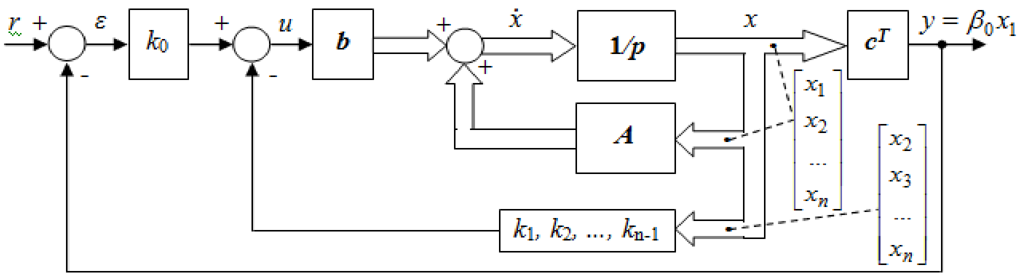

Fig. 2. The structural block scheme of the control system for the object with inertia and astatism.

an integrator element, which increase the order of the designed system (Fig. 3) [2, 6]. The control algorithm, in this case, is determined by the following expression

$$u(t) = -[k_1 \quad k_2 \quad \cdots \quad k_n]\cdot[x_1 \quad x_2 \quad \ldots \quad x_n]^T + k_0\varepsilon . \qquad (10)$$

The characteristic polynomial of the state matrix of the control system is presented:

- for the object with inertia and astatism (Fig. 2)

$$\varphi_A(p) = p^n + \alpha_{n-1}p^{n-1} + \ldots + \alpha_1 p ; \qquad (11, a)$$

- for the object with inertia (Fig. 3)

$$\varphi_{\hat{A}}(p) = p^{n+1} + \alpha_{n-1}p^n + \ldots + \alpha_1 p^2 + \alpha_0 p . \qquad (11, b)$$

The step response of control system will be aperiodic, if the imaginary parts of the characteristic polynomial roots are null. Therefore, in accordance with the method in [3], it is introduced the notion of the maximum stability degree J and the roots of the characteristic polynomial $p_i = -J$. In this case the desired characteristic polynomial $\varphi_c(p)$, obtained by decomposing the polynomial (11) in the linear factors gets the next form:

- for the model object with inertia and astatism

$$\begin{aligned}\varphi_c(p) &= (p+J)^n = c_n p^n + c_{n-1}Jp^{n-1} + \\ &+ c_{n-2}J^2p^{n-2} + \ldots + c_1J^{n-1}p + c_0J^n = \\ &= q_n p^n + q_{n-1}p^{n-1} + q_{n-2}p^{n-2} + \ldots + q_1 p + q_0 ,\end{aligned} \qquad (12, a)$$

where $q_i = f_i(c_i, J),\ i = (0,\ldots,n)$ and value of the maximum stability degree J of the design system is determined by the following expression

$$J = \frac{\alpha_{n-1}}{n} ; \qquad (13, a)$$

- for the model object with inertia

$$\begin{aligned}\varphi_c(p) &= (p+J)^{n+1} = c_{n+1}p^{n+1} + c_nJp^n + \\ &+ c_{n-1}J^2p^{n-1} + \ldots + c_1J^np + c_0J^{n+1} = \\ &= q_{n+1}p^{n+1} + q_np^n + q_{n-1}p^{n-1} + \ldots + q_1p + q_0 ,\end{aligned} \qquad (12, b)$$

where $q_i = f_i(c_i, J),\ i = (0,\ldots,(n+1))$; $q_i = f_i(c_i, J)$, $i = (0,\ldots,n)$ and the value of the maximum stability degree is

$$J = \frac{\alpha_{n-1}}{n+1} . \qquad (13, b)$$

From the characteristic polynomials $\varphi_A(p)$ (11, *a*), $\varphi_c(p)$ (12, *a*) and relations (13, *a*), (1, *b*) for the object with inertia and astatism is obtained

$$J = \frac{\alpha_{n-1}}{n},\ k_0 = q_0/\beta_0;\ k_i = q_i - \alpha_i,\ i = 1,\ldots,(n-1) \qquad (14, a)$$

or from $\varphi_{\hat{A}}(p)$ (11, *b*), $\varphi_c(p)$ (12, *b*) and relations (13,*b*), (1, *b*) for the model object with inertia is obtained

$$J = \frac{\alpha_{n-1}}{n+1},\ k_0 = q_0/\beta_0;\ k_i = q_i - \alpha_{i-1},\ i = (1,\ldots,n) . \qquad (14, b)$$

Using the relations (14) can be calculated the maximum stability degree J and the tuning parameters k_i of the modal controller.

In the expressions (12, *a*, *b*), the relations

$$(p+J)^n = c_n p^n + c_{n-1}Jp^{n-1} + c_{n-2}J^2p^{n-2} + \ldots + c_1J^{n-1}p + c_0J^n \qquad (15, a)$$

$$(p+J)^{n+1} = c_{n+1}p^{n+1} + c_nJp^n + c_{n-1}J^2p^{n-1} + \ldots + c_1J^np + c_0J^{n+1} \qquad (15, b)$$

represent the Newton binomial and their coefficients are calculated by the following expressions [8]

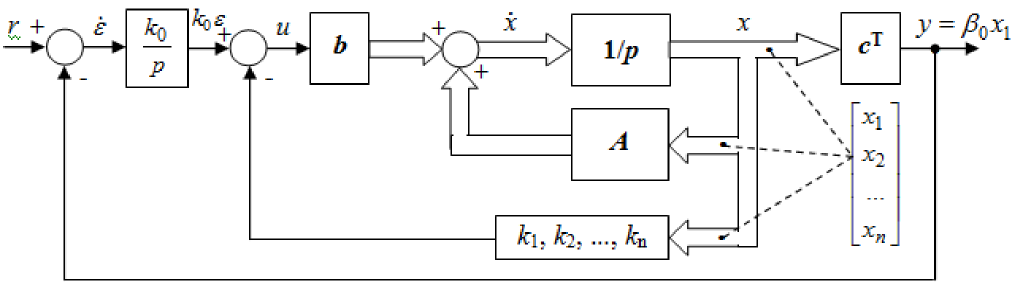

Fig. 3. The structural block scheme of the control system for the object with inertia.

$$c_0 = 1,\ c_i = c_n^i = \frac{n(n-1)...(n-i+1)}{1 \cdot 2 ... i} = \frac{\prod_{l=1}^{i}(n-l+1)}{i!} = \frac{n!}{i!(n-i)!},\ i = (1,...,n) \qquad (16)$$

where for the (15, *a*) have $c_i = c_n^i$ and for the (15, *b*) - $c_i = c_{n+1}^i$.

Using the expressions (14), expression for calculation of the binomial coefficients (16) and taking into account the order of the closed loop system (for the control object with inertia and astatism is *n*, but for the control object with inertia is (*n*+1)), after some transformations were elaborated the analytical algorithms of synthesis the modal controllers, in form of algebraic expressions, for the control object with arbitrary order inertia *n* and with or without astatism for the control system with maximum stability degree and aperiodic step response. The elaborated algorithms are presented in the Tables I and II.

TABLE I.
THE ANALYTICAL ALGORITHMS FOR SYNTHESIS OF THE MODAL CONTROLLERS TO THE OBJECTS WITH INERTIA AND ASTATISM

No.	The calculation expressions
1	Transfer function of the control object $H_F(s) = \dfrac{k}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_{n-2} s^2 + a_{n-1} s}$
2	Normalized transfer function $H_F(s) = \dfrac{\beta_0}{s^n + \alpha_{n-1} s^{n-1} + \alpha_{n-2} s^{n-2} + ... + \alpha_2 s^2 + \alpha_1 s}$, $\alpha_1 = \dfrac{a_{n-1}}{a_0}; ..., \alpha_{n-2} = \dfrac{a_2}{a_0}; \alpha_{n-1} = \dfrac{a_1}{a_0}; \beta_0 = \dfrac{k}{a_0}$.
3	Mathematical model in the vector-matrix form $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$ $y = [\beta_0 \quad 0 \quad 0 \quad ... \quad 0] x.$
4	Controllability condition $rangU = rang[b \quad Ab \quad A^2 b \quad A^3 b ... \quad A^{n-1} b] = n$
5	Control low $u = -[k_1, k_2, k_3, ..., k_{n-1}] \cdot [x_2, x_3, x_4, ..., x_n]^T + k_0 \varepsilon$.
6	Determination of the maximum stability degree and the coefficients of tuning parameters vector $J = \dfrac{\alpha_{n-1}}{n} = \dfrac{a_1}{n a_0}$, $k_0 = \dfrac{J^n}{\beta_0}$, $k_i = \dfrac{\prod_{l=1}^{i}(n-l+1)}{i!} J^{n-i} - \alpha_i$, $i = 1,...,(n-1)$

TABLE II.
THE ANALYTICAL ALGORITHMS FOR SYNTHESIS OF THE MODAL CONTROLLERS TO THE OBJECTS WITH INERTIA

No.	The calculation expressions
1	Transfer function of the control object $H_F(s) = \dfrac{k}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_{n-1} s + a_n}$.
2	Normalized transfer function $H_F(s) = \dfrac{\beta_0}{s^n + \alpha_{n-1} s^{n-1} + \alpha_{n-2} s^{n-2} + ... + \alpha_1 s + \alpha_0}$, $\alpha_0 = \dfrac{a_n}{a_0}; \alpha_1 = \dfrac{a_{n-1}}{a_0}; ..., \alpha_{n-2} = \dfrac{a_2}{a_0}; \alpha_{n-1} = \dfrac{a_1}{a_0}; \beta_0 = \dfrac{k}{a_0}$.
3	Mathematical model in the vector-matrix form $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u,$ $y = [\beta_0 \quad 0 \quad 0 \quad ... \quad 0] x(t).$
4	Mathematical model in the vector-matrix form of the control system $\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \\ \dot{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-1} & 0 \\ -\beta_0 & 0 & \dots & 0 & 0 \end{bmatrix}}_{\hat{A}} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \varepsilon \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\hat{b}} u + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} r,$ $y(t) = [\beta_0 \quad 0 \quad 0 \quad \dots \quad 0] x,$
5	Controllability condition $rangU = rang[\hat{b}, \hat{A}\hat{b}, \hat{A}^2\hat{b}, \dots, \hat{A}^n \hat{b}] = n + 1,$
6	Control low $u(t) = -[k_1, k_2, ..., k_n] \cdot [x_1, x_2, ..., x_n]^T + k_0 \varepsilon,$ where $\dot{\varepsilon} = (r - y)$.
7	Determination of the maximum stability degree and the coefficients of tuning parameters vector $J = \dfrac{\alpha_{n-1}}{n+1} = \dfrac{a_1}{(n+1) a_0}$, $k_0 = \dfrac{J^{n+1}}{\beta_0}$, $k_i = \dfrac{\prod_{l=1}^{i}(n-l+2)}{i!} J^{n+1-i} - \alpha_{i-1}$, $i = 1,...,n$.

Analyzing the elaborated analytical algorithms in this paper and presented in the Tables I and II it was observed that the dynamic tuning parameters of controllers k_i depend on the known object's parameters and the maximum stability degree J of the system. If, however, J is considered as a free parameter, then in accordance with the expression [2, 10]

$$t_s \approx \frac{\ln(1/\varepsilon_{st})}{J} \qquad (17)$$

may be imposed or optimized the settling time of the designed control system. In relation (17) the ε_{st} is the stationary system error.

III. APPLICATIONS AND COMPUTER SIMULATION

Suppose that the controlled technological process is described by the model object with fourth order inertia with known parameters

$$H_F(s) = \frac{6}{(0,5s+1)(s+1)(2s+1)(4s+1)} = \frac{6}{4s^4+15s^3+17,5s^2+7,5s+1}, \quad (18)$$

where $k = 6$, $a_0 = 4, a_1 = 15, a_2 = 17,5, a_3 = 7,5, a_4 = 1$.

It is formulated the problem to synthesis of the modal controller to the model object (18) and to determine the vector of tuning parameters *k*.

It is obtained the normalized transfer function by the a_0

$$H_F(s) = \frac{\beta_0}{s^4+\alpha_3 s^3+\alpha_2 s^2+\alpha_1 s+\alpha_0} = \frac{1,5}{s^4+3,75s^3+4,375s^2+1,875s+0,25},$$

where $\alpha_0 = \frac{a_4}{a_0} = 0,25; \alpha_1 = \frac{a_3}{a_0} = 1,875; \alpha_2 = \frac{a_2}{a_0} = 4,375;$

$$\alpha_3 = \frac{a_1}{a_0} = 3,75; \beta_0 = \frac{k}{a_0} = 1,5.$$

It is determined the vector-matrix equation in the standard controllable realization

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0,25 & -1,875 & -4,375 & -3,75 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$
$$y(t) = \begin{bmatrix} 1,5 & 0 & 0 & 0 \end{bmatrix} x.$$

The stationary error of the control system will be null if in the structure of controller is connected an integrator element (Fig. 3), which raises the order of the designed system and the above equation is transformed in the following form [2, 6]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{\varepsilon} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0,25 & -1,875 & -4,375 & -3,75 & 0 \\ -1,5 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\hat{A}} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \varepsilon \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\hat{b}} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r,$$
$$y = \begin{bmatrix} 1,5 & 0 & 0 & 0 & 0 \end{bmatrix} x.$$

It is obtained the characteristic polynomial of the $\hat{A}$ matrix

$$\varphi_{\hat{A}}(p) = p^5 + 3,75p^4 + 4,375p^3 + 1,875p^2 + 0,25p = p^5 + \alpha_3 p^4 + \alpha_2 p^3 + \alpha_1 p^2 + \alpha_0 p$$

It is verified the condition of controllability of the system

$$rangU = rang[\hat{b}, \hat{A}\hat{b}, \hat{A}^2\hat{b}, \hat{A}^3\hat{b}, \hat{A}^4\hat{b}] = 5.$$

Because the rank of matrix U is equal with order of the system, then the system is controllable.

In conformity with (10) the control algorithm is presented in the following form

$$u(t) = -[k_1, k_2, k_3, k_4] \cdot [x_1, x_2, x_3, x_4]^T + k_0 \varepsilon.$$

Synthesis of controller by the maximum stability degree criterion is done based on the elaborated analytical algorithms, presented in the Table II.

$$J = \frac{\alpha_3}{5} = \frac{a_1}{5a_0}, k_0 = \frac{J^5}{\beta_0}, k_1 = 5J^4 - \alpha_0,$$
$$k_2 = 10J^3 - \alpha_1, k_3 = 10J^2 - \alpha_2, k_4 = 5J - \alpha_3.$$

The results of synthesis are given in the Table III.

Using the method of dominant poles for the system with imposed indices of performance $\sigma < 5\%, t_r < 10s$, it is determined the dominates poles [1, 2, 6]:

- $\sigma < 5\% \Rightarrow \psi \approx 1$;
- $t_r < 10s \Rightarrow 10 \approx \frac{3}{\psi\omega_n} \Rightarrow \eta = \psi\omega_n = 0,3$,

where ω_n is a proper frequency;

- in final is choice $\gamma_{1,2} = p_{dom} = -0,3$, which allows to impose the other poles $\gamma_{3,4,5} = -1$.

The characteristic polynomial of the designed control system is determined by the expression

$$\varphi_c(p) = (p+0,3)^2(p+1)^3 = p^5 + 3,6p^4 + 4,89p^3 + 3,07p^2 + 0,87p + 0,09 = p^5 + q_4 p^4 + q_3 p^3 + q_2 p^2 + q_1 p + q_0.$$

Using the relation (14, *b*) is determined the vector of tuning parameters k, in conformity with the dominant poles method, which values are presented in the Table III.

To calculate the tuning parameters by the parametric optimization method was used the Matlab Simulink software (the simulation structural block scheme of the control system in the state space is presented in the Fig. 4) and the obtained results are presented in the Table III.

TABLE III.
THE RESULTS OF SYNTHESIS THE CONTROLLER

No.	Synthesis methods	k_0	k_1	k_2	k_3	k_4
1	Maximum stability degree	J=0,75				
		0,158	1,33	2,344	1,25	0
2	The domination poles	0,06	0,62	1,195	0,515	-0,15
3	Parametrical optimization	0,091	0,538	0,257	-0,962	-0,915

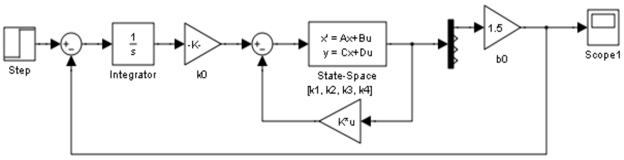

Fig. 4. The simulation structural block scheme of the control system in the state space.

Step responses of the designed control system are presented in the Fig. 5 and the performance are given in the Table IV. The numbering of curves correspond to the numbering of the methods in the Table III.

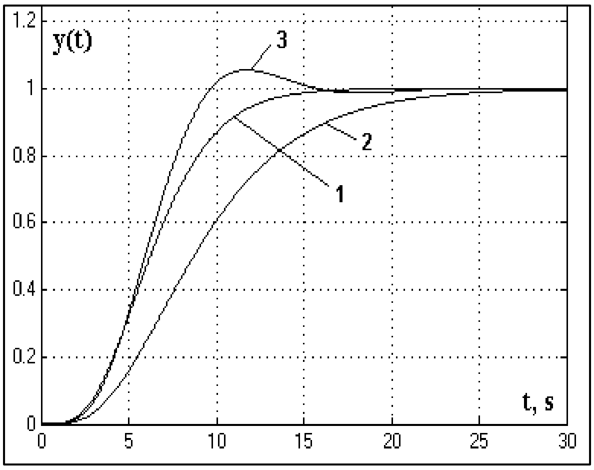

Fig.5. Step responses of the designed control system.

TABLE IV.
THE PERFORMANCE OF THE DESIGNED CONTROL SYSTEM

No.	Synthesis method	The performance of control system				
		t_c, s	t_r, s	σ, %	λ	ψ
1	Maximum stability degree	8,95	12,2	-	-	-
2	The domination poles	15	19,2	-	-	-
3	Parametrical optimization	5,6	12,6	5,5	1	1

Next it is formulated the problem of optimization the settling time of the designed control system. This problem can be solved, because the analytical exprressions for calculation of the components of the feedback vector k_i, elaborated in this paper (view Tables II and III) represent the dependencies by the maximum stability degree J of the designed system and also it is inversely proportional to the settling time t_s (17).

In the result, using the analytical algorithms presented in the Table II were calculated the tuning parameters k_i of the modal controller, that is tuning to the model object (18), for the diverse values of the stability degree J of designed system (Table V).

TABLE V.
CONTROLLERS' SYNTHESIS FOR OPTIMISATION OF t_s

No.	J	k_0	k_1	k_2	k_3	k_4	t_s, s
1	J=0,75	0,158	1,33	2,344	1,25	0	12,2
2	J=1	0,667	4,75	8,125	5,625	1,25	9,15
3	J=1,5	5,063	25,063	31,88	18,13	3,75	6,1
4	J=2	21,33	79,75	78,13	35,63	6,25	4,58

The results of the synthesis are presented in the Fig. 6. The numbering of the curves corresponds to the numbering from the Table V.

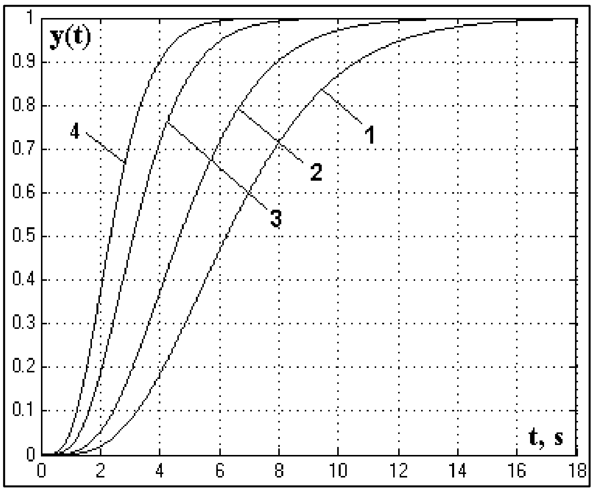

Fig. 6. Step responses of the automatic control system for various values of the settling time t_s.

From Figure 6 it is observed that with increasing the stability degree of the designed control system is decreasing the settling time (view Table V).

IV. CONCLUSIONS

1. In this paper were proposed and theoretic argued the new analytical algorithms, in form of algebraic expressions for synthesis of the modal controllers by the maximum stability degree criterion, that offers to the designed control system higher speed (the lowest settling time), reduced overshoot and better robustness at the variation parameters of the control object.

2. The elaborated algorithms not impose any restrictions to the complexity of the control object and can be applied for different types of objects: with arbitrary order of inertia; inertia and astatism; inertia and time delay; inertia, astatism and time delay. At the same time, the elaborated analytical algorithms permit to impose or to optimize the settling time of the designed control system.

3. Application of the elaborated analytical algorithms eliminates the biggest part of steps provided by the algorithm of the maximum stability degree method. This fact essential simplifies the procedure of synthesis the controller and the volume of work and duration of the processing respectively are reduced approximately with 80-90%. As a result, it frees the resources of computer system, making the control system more responsive to the negative influence of disturbances, which provides the necessary conditions for synthesis and implementation of the controllers with auto-tuning and adaptive control systems.

4. Analyzing the performance of the designed control system by the proposed analytical algorithms for synthesis of the modal controller, in comparison with parametrical optimization and dominant poles methods, it was noticed that elaborated algorithms offer to the designed control systems the aperiodic step response, higher performance and better robustness. At the same time, from the

presented example it can be observed that by the varying the stability degree of the control system it can be imposed and optimized the settling time of the control system.

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