A criterion for estimating the favoring of beneficiaries in apportionments

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Abstract

There is proposed a criterion to quantitatively estimate the favoring of large or small beneficiaries in particular apportionments and on the whole (on an infinity of cases) by apportionment methods. By computer simulation, it is shown that the favoring of beneficiaries by d'Hondt method can be considerable, in total overpassing 10 of apportioned entities.

Keywords: quantitative criteria, favoring a beneficiary, favoring large or small beneficiaries, computer simulation.

1. Introduction

Integer character of the proportional apportionment (APP) problem usually causes a certain disproportion of the representation of deciders' will in the decision [1, 2], some beneficiaries (parties, states, schools, etc.) being favored at the expense of others. Therefore, reducing the favoring in question is one of the basic requirements when choosing the APP method to be applied under concrete situations (bias condition [1, 3]). But there are also other aspects (see, par example, [3, p. 9]), which eventually led to the application of various APP methods, such as Hamilton (Hare), Jefferson, Webster, d'Hondt, Sainte-Laguë, Huntington-Hill [1, 2] and others.

So, when selecting an APP method, the property of non-favoring of beneficiaries is useful. It is well-known that d'Hondt method favors large beneficiaries, and Huntington-Hill method favors the small ones [1, 2]. But which of the two favors beneficiaries to a larger extent? Namely, criteria to investigate such aspects are examined in this paper.

2. Preliminary considerations

The departure point of APP methods is to minimize the disproportion of apportionment of entities (seats, computers, tickets, etc. – a-entities) to beneficiaries (parties, states, schools, etc.). In order to estimate this disproportion, various indices were proposed. Starting from the value *d* of a decider will ($d = M/V$, where M is the total number of a-entities and V is the total number of deciders; *d* reflects unequivocally the rights of each decider in the decision), and basing on a comparative multi-aspectual analysis of 12 indices, in [2] the opportunity of using the Average relative deviation index *I* for this purpose is argued:

$$I = 100\sum_{i=1}^{n}\left|\frac{V_i}{V} - \frac{x_i}{M}\right| \to \min. \qquad (1)$$

Here n is the number of beneficiaries, V_i is the number of deciders and x_i is the number of a-entities apportioned to beneficiary *i*.

Evidently, a beneficiary *i* is considered larger than a beneficiary *k*, if $V_i > V_k$.

To quantitatively estimate the favoring of beneficiaries, the formalization of the notion of beneficiaries favoring is needed. It will be distinguished three notions of favoring of beneficiaries by an APP method:

a) favoring of a beneficiary in an apportionment;
b) favoring of large or small beneficiaries in an apportionment;
c) favoring of large or small beneficiaries on the whole, on an infinity of apportionments.

Also, each of the specified above three aspects can be characterized by:

A) identifying the fact of favoring;
B) quantitatively estimating the favoring of beneficiaries.

Unfortunately, all quantitative criteria, along with the respective quantitative assessments (aspect B), can be used also to identify the fact of favoring of large or small beneficiaries by an APP method (aspect A).

3. Formalizing the notion of favoring of a beneficiary

In case of proportional apportionment, knowing the value of *d*, it is easy to determine the expected rights D_i of beneficiary *i* in the decision, namely, $D_i = dV_i$. In other words, D_i is the influence power of beneficiary *i* in the decision, delegated to it by their V_i deciders. Let's transform (1) as follows

$$I = 100\sum_{i=1}^{n}\left|\frac{V_i}{V} - \frac{x_i}{M}\right| = \frac{100}{M}\sum_{i=1}^{n}\left|dV_i - x_i\right| = \frac{100}{M}\sum_{i=1}^{n}\left|D_i - x_i\right| \to \min. \quad (2)$$

So, if $x_i = D_i$, $i = \overline{1,n}$, we have $I = 0$, that is if the number of a-entities, apportioned to each beneficiary i, is equal to the expected value D_i for $i = \overline{1,n}$, then $I = 0$ and there are no favored beneficiaries. The disproportion in an apportionment can occur because D_i is a real number, and x_i is an integer.

Definition 1 [2]. In an apportionment, a beneficiary i is favored, if it gets an excess of a-entities ($\Delta D_i = x_i - D_i > 0$); is disfavored, if it obtains a deficit number of a-entities ($\Delta D_i < 0$); and is neutral (neither favored nor disfavored), if it gets a number of a-entities equal to the expected one ($\Delta D_i = 0$).

So, the A aspect, for a beneficiary i:

1) favored, occurs $x_i > a_i$, where $a_i = \lfloor D_i \rfloor \leq D_i$;
2) disfavored, occurs $x_i \leq a_i$ at $D_i > a_i$;
3) neutral, occurs $x_i = a_i$ at $D_i = a_i$.

Aspect B for a beneficiary i is characterized by the number of a-entities in excess in the apportionment: $\Delta D_i = x_i - D_i$; unfortunately, if $\Delta D_i < 0$, then the beneficiary i is disfavored, because it has a deficit of a-entities.

It remains to define the other two notions of favoring of large or small parties in an apportionment and on the whole by an APP method – cases (b) and (c) from Section 2.

4. Formalizing the notion of favoring of large/small beneficiaries

First of all, it is useful to mention that, because of $D_1 + D_2 + \ldots + D_n = M$ and $x_1 + x_2 + \ldots + x_n = M$, if some beneficiaries are favored, the other ones are mandatory disfavored. Evidently, there are no alternatives and it is easy to formalize the notion of favoring of large or small beneficiaries if $n = 2$. But there is not the case for $n > 2$. One of the well-known alternatives is done in [1].

Definition 2 (according to [1, p. 125]). An apportionment method favors large parties if

$$\frac{\sum_{i\in L} x_i}{\sum_{i\in L} V_i} > \frac{\sum_{j\in S} x_j}{\sum_{j\in S} V_j} \quad (3)$$

and it favors small parties if

$$\frac{\sum_{i \in L} x_i}{\sum_{i \in L} V_i} < \frac{\sum_{j \in S} x_j}{\sum_{j \in S} V_j}, \tag{4}$$

where L and S are subsets of $\{1, 2, \ldots, n\}$ such that $x_i > x_j$ whenever $i \in L$ and $j \in S$ [3].

Without diminishing the universality of the approach, further it is considered that n beneficiaries are ordered in non-ascending order of V_i, $i = \overline{1, n}$, i.e. $V_1 \geq V_2 \geq V_3 \geq \ldots \geq V_n$. Let's consider the apportionments for which $x_1 > x_2 > x_3 > \ldots > x_n$. For such an apportionment and $|L| + |S| = n$, there are $n - 1$ variants of different pairs of subsets L and S: $L_1 = \{1\}$, $S_1 = \{2, 3, \ldots, n\}$; $L_2 = \{1, 2\}$, $S_2 = \{3, 4, \ldots, n\}$; …; $L_{n-1} = \{1, 2, 3, \ldots, n - 1\}$, $S_{n-1} = \{n\}$. In general, in each of subsets L and S there may be beneficiaries with the same value of x. In such a case, the number of variants of different pairs of subsets L and S is larger than $n - 1$, but it is easy to show that examination of only the described above $n - 1$ variants on the subject in question is also sufficient.

If for all mentioned above $n - 1$ variants of different pairs of subsets L and S the relation (4) takes place, then it is simple to decide that large beneficiaries are favored, and vice versa – if the relation (5) takes place. But there may be cases, when for some pairs of subsets L and S the relation (4) takes place, and for the other pairs of subsets L and S the relation (5) occurs. Let's consider the following example.

Example 1. Let $M = 9$, $n = 4$, $V_1 = 500$, $V_2 = 495$, $V_3 = 395$, $V_4 = 390$ and for apportionment the d'Hondt method is applied. Then $Q = 197.7(7)$ and ΔM=3. The other calculations for the apportionment are shown in Table 1.

Table 1. Calculations for the apportionment to Example 1

i	V_i	a_i	$V_i/(a_i + 1)$	$V_i/(a_i + 2)$	Δx_i	x_i	x_i/V_i
1	500	2	166.(6)	125.00	1	3	0.006000
2	495	2	165.00	123.75	0	2	0.004040
3	395	1	197.50	131,(6)	1	2	0.005063
4	390	1	195.00	130.00	1	2	0.005128

The three variants of subsets L and S are: $L_1 = \{1\}$, $S_1 = \{2, 3, 4\}$; $L_2 = \{1, 2\}$, $S_2 = \{3, 4\}$; $L_3 = \{1, 2, 3\}$, $S_3 = \{4\}$. The results of calculations by formulas (4) and (5) for subsets L_j and S_j (j = 1, 2, 3) are systemized in Table 2.

Table 2. Results of calculus for subsets L_j and S_j (j = 1, 2, 3) to Example 1

j	$\sum_{i\in L_j} x_i \,/\, \sum_{i\in L_j} V_i$	Relation	$\sum_{i\in S_j} x_i \,/\, \sum_{i\in S_j} V_i$	Favored beneficiaries
1	0.006000	>	0.004688	large
2	0.005025	<	0.005096	small
3	0.005036	<	0.005128	small

According to the {L_1, S_1} variant, the requirement (4) takes place, but for the {L_2, S_2} and {L_3, S_3} variants the requirement (5) takes place (see Table 2). So, in this case, to decide whether the obtained apportionment favors large or small beneficiaries or it is neutral by using the Definition 2 is not possible.

Therefore, taking into account Definition 1, it is useful also another approach to the subject in question. Let's note $\Delta D_i = x_i - D_i$, $i = \overline{1,n}$. Also, basing on relation (6), it is opportune to redefine the L and S subsets of large and, respectively, small beneficiaries as follows:

$$L = \{1, 2, \ldots, \lfloor n/2 \rfloor\} \tag{5}$$

$$S = \{\lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, n\}, \tag{6}$$

where $x_i \geq x_j$ whenever $i \in L$ and $j \in S$.

According to (5) and (6), one has $|L| = |S| = \lfloor n/2 \rfloor$ and, if n is even, the pair of subsets {L, S} coincides with one of $n - 1$ variants of subsets {L, S} used by Definition 2.

Definition 3. An apportionment favors large beneficiaries, if the summary a-entities in excess, obtained by large beneficiaries (L), is greater than that, obtained by small beneficiaries (S) and vice versa, that is it favors large beneficiaries if $F_{a1} > 0$, it favors the small ones if $F_{a1} < 0$ and it is neutral if $F_{a1} > 0$, where

$$F_{a1} = \sum_{i=1}^{\lfloor n/2 \rfloor} \Delta D_i - \sum_{i=\lceil n/2 \rceil+1}^{n} \Delta D_i. \tag{7}$$

In addition to identifying the favoring of large or small beneficiaries in an apportionment (aspect A), criterion F_{a1} also allows quantitative estimation of absolute favoring in question, measured in a-entities (aspect B).

When applying (7) to Example 1, one has $F_{a1} = 3 - 500d + 2 - 495d - (2 - 395d + 2 - 390d) = 1 - 210d = 1 - 210 \times 9/1780 = -0.0618$ a-entities

< 0. Thus, according to Definition 3, the apportionment of Example 1 favors small beneficiaries.

To mention that, according to Table 2, if to take into account only the pairs (5) and (6) ($L = L_2 = \{1, 2\}$ and $S = S_2 = \{3, 4\}$), the apportionment of Example 1 also favors small parties. But it is easy to show that conditions (3) and (4), when using the interpretation of pair $\{L_2, S_2\}$ as defined by (5) and (6), are not equivalent to stipulations of Definition 3; they are not interchangeable even for the identification or the fact of favoring of large or small beneficiaries in a particular apportionment.

Moreover, with refer to Definition 2, there are particular apportionments, for which the inequality (4) takes place for all the $n - 1$ variants of subsets L_j and S_j (see Example 2), but on the whole (on an infinity of apportionments) the method favors large beneficiaries and vice versa, the inequality (4) takes place for all the $n - 1$ variants of subsets L_j and S_j (see Example 3), but on the whole (on an infinity of apportionments) the method favors small beneficiaries. So, Definition 2 refers to the favoring of beneficiaries (parties) in particular apportionments.

Example 2 [2]. D'Hondt method favors small beneficiaries. Let $M = 9$, $n = 2$, $V_1 = 500$ and $V_2 = 390$. Then $Q \approx 98.9$ and $\Delta M = 1$. The other results of calculations are shown in Table 3.

Table 3. Calculations to Example 2

i	V_i	a_i	$V_i/(a_i + 1)$	Δx_i	x_i	Δx_i	x_i/V_i
1	500	5	83,3	0	5	0	0.0100
2	390	3	97,5	1	4	1	0.0103

We have $x_1 > x_2$ and $x_1/V_1 = 5/500 = 0.0100 < x_2/V_2 = 4/390 \approx 0.0103$. So, in this apportionment, according to relations (3) and (4), d'Hondt method favors the small beneficiaries (beneficiary 2). The same result is obtained following the stipulations of Definitions 1 and 3. Thus, even the d'Hondt method, which is considered to be strongly favoring large beneficiaries, sometimes favors small beneficiaries.

Example 3 [2]. Huntington-Hill method favors large beneficiaries. Let $M = 26$, $n = 2$, $V_1 = 1000$ and $V_2 = 900$. Then $Q \approx 73.08$, $q^* = 73$ and $\Delta M = 1$. The other results of calculations are shown in Table 4, where $z_i = \lfloor V_i/q^* \rfloor$.

Table 4. Calculations to Example 3

i	V_i	a_i	V_i/q^*	Relation	$\sqrt{z_i\,(z_i+1)}$	x_i	x_i/V_i
1	1000	13	13.70	>	13.49	14	0.014
2	900	12	12.33	<	12.49	12	0.01(3)

We have $x_1 > x_2$ and $x_1/V_1 = 14/1000 = 0.014 > x_2/V_2 = 12/900 = 0.013(3)$. So, in this apportionment, according to relations (3) and (4), Huntington-Hill method favors the large beneficiaries (beneficiary 1). The same result is obtained following the stipulations of Definitions 1 and 3. Thus, even the Huntington-Hill method, which is considered to slightly favoring small beneficiaries, sometimes favors the large ones.

Thus, we have to distinguish between the favoring of beneficiaries in a particular apportionment and the favoring of beneficiaries by an apportionment method on the whole. It can happen that in particular apportionments the method favors large beneficiaries (par example, d'Hondt method), but on the whole, on an infinity of apportionments, it favors small beneficiaries and vice versa (par example, Huntington-Hill method). At the same time, when grouping n parties in subsets L (large beneficiaries) and S (small beneficiaries) according to (5) and (6), the comparative value, in pairs ("larger", "smaller"), of the beneficiaries' number of deciders is taken into account. Therefore, in particular cases it can happen that $V_{\lfloor n/2 \rfloor} > V_{\text{av}i}$ or $V_{\lceil n/2 \rceil+1} < V_{\text{av}i}$, where $V_{\text{av}i} = (V_1 + V_2 + V_3 + \ldots + V_n)/n$. But, in case of an infinity of apportionments and uniform distribution of values V_i, $i = \overline{1,n}$, relations $\text{avrg}\{V_{\lfloor n/2 \rfloor}\} < V_{\text{av}i}$ and $\text{avrg}\{V_{\lceil n/2 \rceil+1}\} > V_{\text{av}i}$ take place.

Definition 4. An apportionment method favors large beneficiaries, if the average summary number of a-entities in excess, obtained by large beneficiaries (L), is greater than that obtained by small beneficiaries (S), and vice versa, that is, it favors large beneficiaries if $\overline{\mathrm{F_{a1}}} > 0$, it favors the small ones if $\overline{\mathrm{F_{a1}}} < 0$ and it is neutral if $\overline{\mathrm{F_{a1}}} = 0$, where $\overline{\mathrm{F_{a1}}}$ is the average of $\mathrm{F_{a1}}$ on an infinity of apportionments.

So,

$$\overline{\mathrm{F}_{a1}} = \lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \left(\sum_{i=1}^{\lfloor n/2 \rfloor} \Delta D_{ik} - \sum_{i=\lceil n/2 \rceil+1}^{n} \Delta D_{ik}\right) = \sum_{i=1}^{\lfloor n/2 \rfloor} \overline{\Delta D_i} - \sum_{i=\lceil n/2 \rceil+1}^{n} \overline{\Delta D_i}\,, \qquad (8)$$

where $\overline{\Delta D_i}$ is the average of ΔD_i on an infinity of apportionments ($K \to \infty$). Criterion $\overline{\mathrm{F}_{a1}}$ is measured in a-entities.

In addition to identifying the APP method favoring of large or small beneficiaries (aspect A), criterion $\overline{\mathrm{F}_{a1}}$ also allows the quantitative estimation of the absolute favoring in question, measured in a-entities (aspect B).

5. A case study: favoring of beneficiaries by d'Hondt method

To determine the value of criterion $\overline{\mathrm{F}_{a1}}$, computer simulation with the SIMAP application was used. The initial data for calculations are: M = 6, 11, 21, 51, 101, 201, 501; n = 2, 3, 4, 5, 7, 10, 15, 20, 30, 50; $n \le M - 1$; $V = 10^8$; uniform distribution of values V_i, $i = \overline{1, n}$; sample size 10^6. So, we have 58 variants of values for the pair {M, n}: 4 + 6 + 8 + 10 × 4 = 58. The graphs of criterion $\overline{\mathrm{F}_{a1}}$ dependence to M and n, when using the d'Hondt method, are presented in Figure 1.

From Figure 1 it can be seen that $\overline{\mathrm{F}_{a1}}$(d'H) value is increasing to n and slightly increasing to M, especially at $M \ge 2n$. For $6 \le M \le 501$, $2 \le n \le 50$ and $n < M$, the $\overline{\mathrm{F}_{a1}}$(d'H) value at n = 2 is in the range of 0.32÷0.41 a-entities, and at n = 50 it is in the range of 9.0÷12.1 a-entities, being considerable, especially at relatively high values of n.

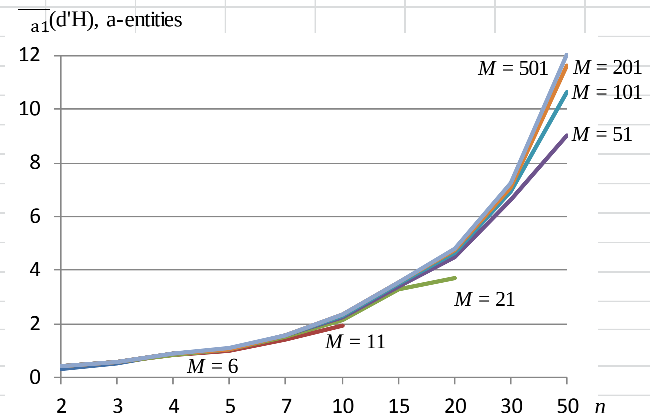

Figure 5. Criterion $\overline{\mathrm{F}_{a1}}$ dependence to M and n for d'Hondt method.
Figure 1. Criterion $\overline{\mathrm{F}_{a1}}$ dependence to M and n.

5. Conclusion

There is a clear distinction between favoring of beneficiaries in an apportionment and favoring of beneficiaries on the whole by an APP method. The proposed criteria F_{a1} and $\overline{F_{a1}}$ (see Definitions 3 and 4) can be used to quantitatively estimate the favoring of large or small beneficiaries in an apportionment or, respectively, on the whole by an APP method. The calculations carried out for the d'Hondt method show that the favoring of large beneficiaries can overpass 10 a-entities.

References

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