Proceedings of the 4th Conference of Mathematical Society of Moldova
CMSM4'2017, June 28-July 2, 2017, Chisinau, Republic of Moldova

On periodic solutions of the four-dimensional differential system of Lyapunov-Darboux type with quadratic nonlinearities

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Abstract

For the four-dimensional differential system of Lyapunov-Darboux type with quadratic nonlinearities, we have found a holomorphic integral of Lyapunov type. Using this integral and the Lyapunov theorem, we have obtained centro-affine invariant conditions for stability of unperturbed periodic motion.

Keywords: Differential system, center-affine comitant, stability of unperturbed motion.

1 Introduction

The differential systems with polynomial nonlinearities are important in various applied problems. One of the interesting cases is differential systems, which characteristic equations have purely imaginary roots. In this paper we consider a four-dimensional differential system, which characteristic equation has two simple purely imaginary roots and the other two roots have real negative parts.

2 Stability of unperturbed periodic motions of Lyapunov-Darboux type differential system with quadratic nonlinearities

Let the characteristic equation of the differential system

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + X^j \quad x^1, x^2, ..., x^{n+2}\Big) \quad (j, \alpha = \overline{1, n+2}) \tag{1}$$

has two purely imaginary simple roots $\lambda\sqrt{-1}$ and $-\lambda\sqrt{-1}$, where X^j are holomorphic functions of x^j $(j = \overline{1,4})$.

The systems (1) with two purely imaginary roots and n roots with negative real part of the characteristic equation will be called *systems of Lyapunov type.*

Consider the system of differential equations

$$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = \overline{1,4}), \tag{2}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices, in which the complete convolution is made and the group of center-affine transformations $GL(4, \mathbb{R})$.

The following center-affine invariant polynomials of the system (2) are known from [3]:

$$I_{1,4} = a^\alpha_\alpha,\ I_{2,4} = a^\alpha_\beta a^\beta_\alpha,\ I_{3,4} = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta,\ I_{4,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta a^\delta_\gamma,\ P_{1,4} = a^\alpha_{\alpha\beta} x^\beta,$$

$$P_{2,4} = a^\alpha_\beta a^\beta_{\alpha\gamma} x^\gamma,\ P_{3,4} = a^\alpha_\gamma a^\beta_\alpha a^\gamma_{\beta\delta} x^\delta,\ P_{4,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta a^\delta_{\gamma\mu} x^\mu,$$

$$S_{0,4} = u_\alpha x^\alpha,\ S_{1,4} = a^\alpha_\beta x^\beta u_\alpha,\ S_{2,4} = a^\alpha_\gamma a^\beta_\alpha x^\gamma u_\beta,\ S_{3,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta x^\delta u_\gamma,$$

$$\overline{R}_{6,4} = a^\alpha_p a^\beta_q a^\gamma_\beta a^\delta_r a^\mu_\delta a^\nu_\mu u_s u_\alpha u_\gamma u_\nu \varepsilon^{pqrs},\ \overline{R}_{6,4} = det\left(\frac{\partial S_{i-1,4}}{\partial x^j}\right)_{i,j=\overline{1,4}},$$

$$K_{6,4} = a^\alpha_\theta a^\beta_\gamma a^\gamma_\varphi a^\delta_\mu a^\mu_\nu a^\nu_\psi x^\theta x^\varphi x^\psi x^\tau \varepsilon_{\alpha\beta\delta\tau},\ \widetilde{K}_{1,4} = a^\alpha_{\beta\gamma} x^\beta x^\gamma y^\delta z^\mu \varepsilon_{\alpha\gamma\delta\mu}, \tag{3}$$

where vectors $y = (y^1, y^2, y^3, y^4)$ and $z = (z^1, z^2, z^3, z^4)$ are cogradient with vector of phase variables $x = (x^1, x^2, x^3, x^4)$, and vector

$u = \ (u_1, u_2, u_3, u_4)$ is covariant with vector x [2]. Polynomials $I_{i,4}$ $(i = \overline{1,4})$ are invariants, $P_{j,4}$ $(j = \overline{1,4})$ and $K_{6,4}$ are comitants, $S_{j,4}$ $(j = \overline{0,3})$ are mixed comitants, $\overline{R}_{6,4}$ is contravariant, and $\widetilde{K}_{1,4}$ is comitant of cogradient vectors x, y, z [2].

Lemma 1 [3]. *If $\widetilde{K}_{1,4}$ from* (3) *is identically equal to zero* ($\widetilde{K}_{1,4} \equiv 0$) *then the system* (2) *takes the form*

$$\dot{x}^j = a^j_\alpha x^\alpha + 2x^j \ \ a^1_{1\alpha} x^\alpha) \quad (j, \alpha = \overline{1,4}). \tag{4}$$

The system (4) is called *four-dimensional differential system of Darboux type.*

Remark 1. *For any centro-affine transformation of the system* (4)*, its quadratic part retains its form changing only the variables and coefficients. This follows from the fact that the identity $\widetilde{K}_{1,4} \equiv 0$ is preserved under any centro-affine transformation.*

Lemma 2 [3]. *If $\overline{R}_{6,4} \not\equiv 0$, then by the centro-affine transformation*

$$\overline{x}^1 = S_{0,4}, \quad \overline{x}^2 = S_{1,4}, \quad \overline{x}^3 = S_{2,4}, \quad \overline{x}^4 = S_{3,4}, \tag{5}$$

the system (4) *can be brought to the following form:*

$$\dot{x}^1 = x^2 + 2x^1 \left(a^1_{1\alpha} x^\alpha\right), \ \dot{x}^2 = x^3 + 2x^2 \left(a^1_{1\alpha} x^\alpha\right), \ \dot{x}^3 = x^4 + 2x^3 \ \ a^1_{1\alpha} x^\alpha\right),$$

$$\dot{x}^4 = -L_{4,4} x^1 - L_{3,4} x^2 - L_{2,4} x^3 - L_{1,4} x^4 + 2x^4 \ \ a^1_{1\alpha} x^\alpha\right), \tag{6}$$

where

$$L_{1,4} = -I_{1,4}, \quad L_{2,4} = \frac{1}{2}\,(I^2_{1,4} - I_{2,4}), \quad L_{3,4} = \frac{1}{6}\,(3I_{1,4}I_{2,4} - 2I_{3,4} - I^3_{1,4}),$$

$$L_{4,4} = \frac{1}{24}\,(8I_{1,4}I_{3,4} - 6I_{4,4} - 6I^2_{1,4}I_{2,4} + 3I^2_{2,4} + I^4_{1,4}), \tag{7}$$

$I_{k,4}$ $(k = \overline{1,4})$ *are from* (3).

Remark 2. *The characteristic equation of the system* (6) *has the form*

$$\rho^4 + L_{1,4}\,\rho^3 + L_{2,4}\,\rho^2 + L_{3,4}\,\rho + L_{4,4} = 0, \tag{8}$$

where $L_{i,4}$ $(i=\overline{1,4})$ *are from* (7).

Lemma 3. *The characteristic equation* (8) *has two simple purely imaginary roots* $\lambda\sqrt{-1}$ *and* $-\lambda\sqrt{-1}$ *and two other roots with real negative part if and only if*

$$L_{1,4}>0, \quad L_{2,4}>0, \quad L_{1,4}L_{2,4}-L_{3,4}>0,$$

$$L_{1,4}^2L_{4,4}+L_{3,4}^2-L_{1,4}L_{2,4}L_{3,4}=0, \tag{9}$$

where $L_{i,4}$ $(i=\overline{1,4})$ *are from* (7).

Theorem 1. *If* $\widetilde{K}_{1,4}\equiv 0$ *and* $\overline{R}_{6,4}\not\equiv 0$*, then under conditions* (9)*, using centro-affine transformation the system* (2) *can be brought to the following form* ($x=x^1$, $y=x^2$, $z=x^3$, $u=x^4$)*:*

$$\dot{x}=-\lambda\, y+2x\psi\equiv P, \quad \dot{y}=\lambda\, x+2y\psi\equiv Q, \quad \dot{z}=u+2z\psi\equiv R,$$

$$\dot{u}=y+(\lambda^2+c)\, z+d\, u+2u\psi\equiv S, \tag{10}$$

where $\lambda=\pm\sqrt{\frac{L_{3,4}}{L_{1,4}}}$ $(L_{1,4}L_{3,4}>0)$, $c=-L_{2,4}$, $d=-L_{1,4}$, $L_{i,4}$ *are from* (7), $\psi=Ax+By+Cz+Du$ *with* A, B, C, D *real constants.*

The system (10) will be called *differential system of Lyapunov-Darboux type.*

Theorem 2. *The functions*

$$\zeta_1=x^2+y^2, \quad \zeta_2=\lambda^3+c\lambda-2(Bc-C+B\lambda^2)\, x+2A(c+\lambda^2)\, y+$$

$$+2\lambda(-Cd+cD+D\lambda^2)\, z+2C\lambda\, u, \quad \zeta_3=\lambda^2\, x^2+d\lambda\, xy+cd\lambda\, xz+$$

$$+\lambda(2c+d^2+4\lambda^2)\, xu-(c+\lambda^2)\, y^2-[2c^2+(6c+d^2)\lambda^2+4\lambda^4]\, yz-$$

$$-cd\, yu-[c^3+c(5c+d^2)\lambda^2+(8c+d^2)\lambda^4+4\lambda^6]\, z^2-$$

$$-[c^2+(4c+d^2)\lambda^2+4\lambda^4] \;\; dzu-u^2\big) \tag{11}$$

are particular integrals of the system (10), $F=\zeta_1\,\zeta_2^{-2}$ *is prime integral of Darboux type of the system* (10).

Proof. Denote by

$$\Lambda = P\,\frac{\partial}{\partial\, x} + Q\,\frac{\partial}{\partial\, y} + R\,\frac{\partial}{\partial\, z} + S\,\frac{\partial}{\partial\, u},$$

the operator of the system (10). By a direct calculation we obtain

$$\Lambda\,(\zeta_1) = 4\,\zeta_1\,\psi, \quad \Lambda\,(\zeta_2) = 2\,\zeta_2\,\psi, \quad \Lambda\,(\zeta_3) = \zeta_3\,(d + 4\psi),$$

$$\Lambda\left(\zeta_1^{\alpha}\,\zeta_2^{\beta}\right) = 2(2\alpha+\beta)\,\zeta_1^{\alpha}\,\zeta_2^{\beta}\,\psi,$$

where $\psi = Ax + By + Cz + Du$. The theorem is proved.

From [3] the following comitant of the system (2) is known:

$$\Phi_{4,4} = L_{4,4} - 2(4/5\,L_{3,4}P_{1,4} + L_{2,4}P_{2,4} + L_{1,4}P_{3,4} + P_{4,4}), \tag{12}$$

where $P_{j,4}$ $(j = \overline{1,4})$ are from (3), $L_{i,4}$ $(i = \overline{1,4})$ are from (7).

Remark 3. *For the system* (10) *for* $x = x^1$, $y = x^2$, $z = x^3$, $u = x^4$ *we have* $\Phi_{4,4} = -\lambda\zeta_2$, *where* ζ_2 *is from* (11).

Remark 4. *The prime integral* $F = \zeta_1\,\zeta_2^{-2}$ *of the system* (10) *with* $\zeta_2 \not\equiv 0$ $(\Phi_{4,4} \not\equiv 0)$ *can be written as a holomorphic Lyapunov integral ([1], §40)* $F = x^2 + y^2 + \widetilde{F}(x,y,z,u)$, *where* $\widetilde{F}(x,y,z,u)$ *contains terms of degree at lest two in variables* x, y, z, u.

Using the Lyapunov theorem ([1], p.160), lemmas 1–3, theorems 1–2 and remarks 3–4, we obtain

Theorem 3. *Assume for the system* (2) *with* $\widetilde{K}_{1,4} \equiv 0$ *and* $\overline{R}_{6,4} \not\equiv 0$ *under centro-affine invariant conditions* (9), *the comitant* (12) *is not identically zero. Then the system has a periodic solution containing an arbitrary constant, and varying this constant one can obtain a continuous sequence of periodic motions, which comprises the studied unperturbed motion. This motion is stable and any perturbed motion, sufficiently close to the unperturbed motion, will tend asymptotically to one of the periodic motions.*

3 Conclusion

Using of centro-affine invariants and comitants of the four-dimensional differential system with quadratic nonlinearities we obtain extension of the results stated in the Lyapunov theorem ([1], §40) concerning the stability of unperturbed periodic motion of the studied system.

Acknowledgments. The project 15.817.02.03F from Supreme Council for Science and Technological Development of the Academy of Sciences of Moldova has supported part of the research for this paper.

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