Proceedings of the 4th Conference of Mathematical Society of Moldova
CMSM4'2017, June 28-July 2, 2017, Chisinau, Republic of Moldova

Invariant conditions of stability of unperturbed motion for differential systems with quadratic nonlinearities in the critical case

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Abstract

The center-affine invariant conditions of stability of unperturbed motion governed by differential systems in the plane with quadratic nonlinearities in the critical case were determined.

Keywords: Differential systems, stability of unperturbed motions, center-affine comitants and invariants, Sibirsky graded algebras.

1 Introduction

Differential systems with polynomial nonlinearities play an important role in practical problems. Among them, the more spread are the Lyapunov critical systems, i.e. the systems with one root of the characteristic equation equal to zero and the others roots with negative real parts. In this paper the systems with quadratic nonlinearities of the Lyapunov form are studied.

2 Stability of unperturbed motion

We examine the differential system with quadratic nonlinearities

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \quad (j, \alpha, \beta = 1, 2), \tag{1}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices in which the total convolution is done.

The tensorial forms of generators of the Sibirsky algebras [1] of the system (1) will be written [2]:

$$I_1 = a^\alpha_\alpha,\ \ I_2 = a^\alpha_\beta a^\beta_\alpha,\ \ I_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta}\varepsilon^{pq},\ \ K_1 = a^\alpha_{\alpha\beta}x^\beta,\ \ K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq},$$

$$K_3 = a^\alpha_\beta a^\beta_{\alpha\gamma}x^\gamma,\ \ K_4 = a^\alpha_\gamma a^\beta_{\alpha\beta}x^\gamma,\ \ K_5 = a^p_{\alpha\beta}x^\alpha x^\beta x^q\varepsilon_{pq},\ \ K_7 = a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}x^\gamma x^\delta,$$

$$K_8 = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta}x^\delta,\ \ K_{11} = a^p_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma x^q\varepsilon_{pq},\ \ K_{12} = a^\alpha_\beta a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}x^\delta x^\mu, \tag{2}$$

$$K_{13} = a^\alpha_\gamma a^\beta_{\alpha\beta}a^\gamma_{\delta\mu}x^\delta x^\mu,$$

where $\varepsilon^{pq}(\varepsilon_{pq})$ is the unit bivector with coordinates.

It is easy to show that if the invariant conditions are satisfied

$$I_1^2 - I_2 = 0,\ \ I_1 < 0, \tag{3}$$

then system (1), by a center-affine transformation, can be brought to the following critical system of the Lyapunov form

$$\frac{dx^1}{dt} = a^1_{\alpha\beta}x^\alpha x^\beta,\ \ \frac{dx^2}{dt} = a^2_\alpha x^\alpha + a^2_{\alpha\beta}x^\alpha x^\beta\ \ (\alpha,\beta = 1,2). \tag{4}$$

Remark 1. In this paper the Lyapunov Theorem on stability of unperturbed motion [3, §32] will be called the Lyapunov Theorem.

Let us introduce the following notations

$$P = (a^2_2)^2 a^1_{11} - 2a^2_1 a^2_2 a^1_{12} + (a^2_1)^2 a^1_{22},\ Q = (a^2_2)^2 a^2_{11} - 2a^2_1 a^2_2 a^2_{12} + (a^2_1)^2 a^2_{22},$$

$$R = (a^2_2)^2 a^1_{11} - (a^2_1)^2 a^1_{22},\ S = a^2_1 a^1_{22} - a^2_2 a^1_{12}\quad (I_1 = a^2_2 < 0). \tag{5}$$

Taking into account (5) and the Lyapunov Theorem on stability of unperturbed motion in system (4) we have the following lemma.

Lemma 1. *The stability of unperturbed motion in system* (4) *is described by one of the following six possible cases:*

I. $P \neq 0$, *then the unperturbed motion is unstable;*

II. $P = 0, QS > 0$, *then the unperturbed motion is unstable;*

III. $P = 0, QS < 0$, *then the unperturbed motion is stable;*

IV. $R = S = 0,\ a^1_{22}Q \neq 0$, *then the unperturbed motion is unstable;*

V. $P = Q = 0$, *then the unperturbed motion is stable;*

VI. $a^1_{11} = a^1_{12} = a^1_{22} = 0$, *then the unperturbed motion is stable.*

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motion, moreover in Case (iii) it is also asymptotic stable [4]. The expressions P, Q, R, S *are given in* (5).

Later on, we make use of the following expressions of the invariants and comitants of system (1) given in (2):

$$E_1 = I_1^2 K_1 - I_1(K_3 + K_4) + K_8,$$

$$E_2 = I_1^3(K_1^2 - K_7) + 2I_1^2(K_1K_4 - 2K_1K_3 - K_{13}) + 2I_1(I_5K_2 + 2K_3^2 - K_4^2) +$$

$$+4K_8(K_4 - K_3) + 2I_2K_{12},\ \ E_3 = I_2K_1 + I_1(K_4 - K_3) - K_8,$$

$$E_4 = I_1(K_{11} - K_1K_2) + K_2(K_4 - K_3),\ \ E_5 = K_{11} - I_1K_5. \qquad (6)$$

Theorem 1. *Let for differential system of the perturbed motion* (1) *the invariant conditions* (3) *are satisfied. Then the stability of the unperturbed motion in system (1) is described by one of the following six possible cases:*

I. $E_1 \not\equiv 0$, *then the unperturbed motion is unstable;*

II. $E_1 \equiv 0,\ \ E_2 > 0$, *then the unperturbed motion is unstable;*

III. $E_1 \equiv 0,\ \ E_2 < 0$, *then the unperturbed motion is stable;*

IV. $E_3 \equiv 0, E_4E_5 \not\equiv 0$, *then the unperturbed motion is unstable;*

V. $E_4 \equiv 0$, *then the unperturbed motion is stable;*

VI. $E_5 \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases the unperturbed motion belongs to some continuous series of stabilized motion, and moreover in Case III it is also asymptotic stable. The expressions E_i $(i = \overline{1,5})$ *are given in* (6).

Remark 2. The extended conditions for Lyapunov's example [3, §32] are obtained from Theorem 1.

3 Conclusion

In this paper the differential system given in Example 2 [3, §32] was investigated by means of comitants and invariants of the Sibirsky algebras of differential system with quadratic nonlinearities.

Acknowledgements. This research was partially supported by grants 15.817.02.18F, 16.80012.02.01F and 15.817.02.03F.

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