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Apportionment "Population paradox" and the Paradox of population influence

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Abstract

A new approach is grounded with respect to the population paradox (PPr). Further on, the paradox of population influence (PPi) is proposed. It is proven that Hamilton method is immune to the PPi, and that d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë methods – are not. By computer simulation, the percentage of non-immunity of Hamilton method to PPr, and the one of d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë divisor methods to PPi, is estimated. For a large range of initial data, this percentage, in the case of the four investigated divisor methods, does not exceed, on average, 0.6-0.8%, that is one case per a total of 120-170 cases.

Keywords: apportionment method, population paradox, paradox of population influence, computer simulation, comparative analyses.

MSC 2010: 62P25, 91B10, 91B12, 68U20

1 Introduction

The population paradox (PPr) was identified and defined in the early 1900s in the case of Hamilton apportionment method [1-3]. Initially, the apportionment of seats in the US House of Representatives at population growth was found to be inappropriate. Later on, the paradox became one of the key restrictions in the apportionment of elective bodies' mandates among parties based on voting results, and sometimes – in the distribution of discrete identical goods among beneficiaries.

In this paper, the reasonable area of coverage by PPr of multiple situations is investigated and a new approach, covering all possible situations, is proposed. In addition, the frequency of non-immunity to population paradox (in its traditional formulation and in the proposed one) of some well-known apportionment methods is estimated by computer simulation.

2 Aspects of Population paradox

There are many definitions of the Population paradox, but their essence is usually the same, for example:

1) "the population paradox occurs when state A loses a seat to state B even though the population of A grew at a higher rate than the population of B" [3];

2) "The population paradox occurs when, based on updated population figures, a reapportionment of a fixed number of seats causes a state to lose a seat to another state, although the percent increase in the population of the state that loses the seat is bigger than the percent increase in the population of the state that gains the seat" [4].

Thus, if Hamilton method [2, 3] had been used in 1901 to reallocate 386 seats in the US Congress House of Representatives, Virginia, at a higher rate of population increase than that of Maine, would have lost a seat, and Maine would have won a seat (Table 1) [5].

On the basis of formulations from [3, 4], the Population paradox is formalized in Definition 1, where for the first apportionment the following notations are used:

M – total number of seats;

n – number of states. Consider $n \leq M$;

V – total population for n states;

V_i – population of state i, $i = \overline{1, n}$;

x_i – number of seats to be allocated to state i, $i = \overline{1, n}$.

Table 1. The Population paradox for states Maine and Virginia [5]

State	Population		Population growth		Seats	
	1900	1901	rate,%	abs., pers.	1900	1901
Maine	694466	699114	1.0067	4648	3	4
Virginia	1854184	1873951	1.0107	19767	10	9

For the second apportionment, here and further on, the nominated and other notations will be completed with the apostrophe symbol ($'$), for example V' for V.

Definition 1. *The Population paradox (PPr) occurs, if at* $M' = M$, $n' = n$ *and*

$$V'_k/V_k > V'_j/V_j \tag{1}$$

the following relations take place

$$x'_k = x_k - 1, \tag{2}$$
$$x'_j = x_j + 1. \tag{3}$$

In this definition, there are doubts about the correctness of condition (1). Let's consider Example 1.

Example 1. *on Population paradox according to Definition 1 when the Hamilton method is applied. Let* $M = 100$, $n = 4$ *and, for the first apportionment,* $V_1 = 50000$, $V_2 = 1400$, $V_3 = 1300$ *and* $V_4 = 800$, *and for the second one:* $V'_1 = 50800$, $V'_2 = 1421$, $V'_3 = 1326$ *and* $V'_4 = 816$. *Then for the first apportionment we have* $Q = (50000 + 1400 + 1300 + 800)/100 = 535$, *and for the second one* $Q' = (50800 + 1421 + 1326 + 816)/100 = 543.63$. *Here* $Q = V/M$ *is the standard divisor, known also as Hare quota [2, 7]. The results of other calculations are shown in Table 2.*

In Table 2 and further on, the notation $R_i = V_i/Q$ is used, where R_i is the standard quota [3].

Table 2. Results of calculations to Example 1

State i	$Q = 535$			$Q' = 543.16$				
	V_i	R_i	x_i	V_i'	R_i'	x_i'	V_i'/V_i	$V_i' - V_i$
1	50000	93.46	93	50800	93.53	94	1.016	800
2	1400	2.62	3	1400	2.58	3	1	0
3	1300	2.43	2	1300	2.39	2	1	0
4	800	1.50	2	816	1.50	1	1.020	16
Total	53500		100	54316		100		

According to Table 2, the Hamilton solution for the second apportionment is to withdraw a seat from state 4 and reallocate it to state 1, in spite of the fact that the population growth rate of state 4 is 2%, while that of state 1 is 1.6%, i.e. smaller. Thus, according to Definition 1, the population paradox (PPr) would occur.

At the same time, one can mention that, in absolute terms, the population of state 4 has increased by only 16 inhabitants, while that of state 1 — by 800 inhabitants, that is, $800/16 = 50$ times more. Moreover, we have $800 > Q' = 543.16$, that is, at a proportional allocation of seats, more than one seat would correspond to the new population of state 1, while much less than one seat ($16 << Q' = 543.16$) – to the new population of state 4. Under such conditions, one cannot affirm that a paradox occurs. Therefore, not in all cases Definition 1 specifies situations of population paradox. Respectively, it would be appropriate to redefine the paradox situations implied by changes in the number of inhabitants.

3 Redefining the Population paradox

At a first glance, there are several aspects and, respectively, alternatives of comparison of two consecutive apportionments with a view to identifying situations in which population paradox occurs. In addition to the use of "population rate deviation" as a comparison criterion from one apportionment to the next one (Definition 1), the "population absolute deviation” and the “absolute deviation of population influence power” are also discussed under this section as criteria.

3.1 Population absolute deviation as criterion

For the comparing of two consecutive apportionments, the decrease ($x'_k - x_k < 0$, see (2)) or the increase ($x'_j - x_j > 0$, see (3)), for each particular state, of the number of seats x_i from one apportionment to the next one, are used. At the same time, because the distribution of seats among states must be made proportionately to the number of inhabitants (V_i, $i = \overline{1,n}$, and respectively, V'_i, $i = \overline{1,n}$), it should be that the absolute increase (decrease) of the number of seats $x'_i - x_i$ be correlated with the absolute increase (decrease) of population $V'_i - V_i$, and not with its rate V'_i/V_i. That is, when defining the population paradox, instead of condition (1) it would be more appropriate to use the following one:

$$V'_k - V_k > V'_i - V_i. \tag{4}$$

Possibly, it is this approach that is taken into account in Example 2 and the population paradox definition from [6]: “The population paradox occurs when one state loses a seat and another state gains a seat, even though the first state’s population increased more than the second state’s population”.

Example 2. [6] *Let $M = 25$, $n = 3$ (states A, B and C) and, for the first apportionment, $V_A = 13$, $V_B = 12$ and $V_C = 112$, and for the second one: $V'_A = 14$, $V'_B = 12$ and $V'_C = 114$. The results of some calculations, when applying the Hamilton method, are shown in Table 3.*

Data of Table 3 show that state A, the population of which increased by 1 mil, gains a seat and state C, with a 2 mil population increase, loses a seat. Based on these results, in [6] it is concluded that the Population paradox occurs. But this example is not one of Population paradox in sense of Definition 1, given that the population growth rate is of 1.077 in case of state A and of 1.018 – in the case of state C.

Table 3. Calculations for the example from [6], Hamilton method

State	Population, mil		Population growth		Standard quota		Seats	
	App. I	App. 2	Rate	Abs.	App. I	App. 2	App. I	App. 2
A	13	14	1.077	1	2.37	2.50	2	3
B	12	12	1	0	2.19	2.14	2	2
C	112	114	1.018	2	20.44	20.36	21	20
Total	137	140			25	25	25	25

Under condition (4) instead of condition (1), the term of "absolute deviation population paradox" or, shorter, "Absolute population paradox" (PPa) will be used in this paper. Respectively, instead of the traditional term of "Population paradox" (in the meaning of Definition 1), the term "rate deviation population paradox" or, shorter, "Rate population paradox" (PPr) will be used.

Thus, for PPa we have the following definition:

Definition 2. *The Absolute population paradox occurs, if at $M' = M$, $n' = n$ and at constraint (4) the relations (2) and (3) take place.*

The correlation between conditions (1) and (4) of population paradox according to Definition 1 (PPr) and, respectively, Definition 2 (PPa), is of interest.

Statement 1. *The conditions of Definition 1 supplemented with those of*

$$V_k \geq V_j, \qquad (5)$$

fall under the conditions of Definition 2.

Indeed, it is sufficient to prove that, if relations (1) and (5) take place, then relation (4) also occurs. Let the conditions (1) and (5) occur. From (1) we have $V'_k/V_k - 1 > V'_j/V_j - 1$, i.e. $(V'_k - V_k)/V_k > (V'_j - V_j)/V_j$ or $(V'_k - V_k) > (V'_j - V_j)V_k/V_j$. The last inequality, if relation (5) takes place, implies that condition (4) occurs. ■

Considering the data of Table 1, it can be easily seen that the paradox regarding the apportionment of seats between states of Virginia and Maine in 1990 and 1991 falls under both conditions of Statement 1 and those of Definition 2.

It can be expected, at the same time, that there are also cases of compliance with the conditions of Definition 2 and non-compliance with the conditions of Statement 1.

Statement 2. *The area of situations, covered by conditions of Definition 2, is larger than the one covered by conditions of Statement 1.*

Indeed, relations (2) and (3) are common to Definition 2 and Statement 1, and, according to Statement 1, the conditions (1) and (5) imply the inequality (4). ▼

It remains to prove that there are also other situations of PPa covered by conditions (2)-(4) but not covered by inequalities (1)-(3), (5); i.e. not in all cases when inequalities (2)-(4) take place, relations (1)-(3), (5) occur as well. One of such cases is determined by conditions (2)-(4) and inequalities

$$V_k > V_j, \tag{6}$$

$$V'_k/V_k < V'_j/V_j, \tag{7}$$

the last inequality being an opposite of relation (1). It is sufficient to show that this case is a real one.

The compatibility of relations (4)-(6) is evident; also, because of (6), relation (4) can be complied with even under condition (7). As to inequalities (2) and (3), it is easier to comply with them in conditions (6) and (7) than in conditions (1) and (5). ■

Consequence 1. *The conditions of Definition 2 cannot be replaced by those of Statement 1.*

Indeed, according to Statement 2 the conditions of Definition 2 are larger than those of Statement 1. ■

The veracity of Statement 2 is confirmed also by Example 3.

Table 4. Results of calculations to Example 3

State i	$Q = 76.5$			$Q' = 78.52$				
	V_i	R_i	x_i	V_i'	R_i'	x_i'	V_i'/V_i	$V_i' - V_i$
1	3400	44.44	45	3502	44.60	44	1.03	102
2	2000	26.14	26	2100	26.74	27	1.05	100
3	1150	15.03	15	1150	14.65	15	1	0
4	1100	14.38	14	1100	14.01	14	1	0
Total	7650		100	7852		100		

Example 3. *on the Absolute population paradox when applying the Hamilton method. Let* $M = 100$, $n = 4$ *and the first apportionment be characterized by data* $V_1 = 3400$, $V_2 = 2000$, $V_3 = 1150$, $V_4 = 1100$, *and the second one by:* $V_1' = 3502$, $V_2' = 2100$, $V_3' = 1150$, $V_4' = 1100$. *Then for the first apportionment we have* $Q = (3400 + 2000 + 1150 + 1100)/100 = 76.5$, *and for the second one* $Q' = (3502 + 2100 + 1150 + 1100)/100 = 78.52$. *The obtained results are shown in Table 4.*

In Example 3, for states 3 and 4, the population does not change from the first apportionment to the second one (1150 and 1100, respectively); nor changes the number of mandates. But for states 1 and 2 we have: $V_1' - V_1 = 3502 - 3400 = 102$; $V_2' - V_2 = 2100 - 2000 = 100$; $V_1'/V_1 = 3502/3400 = 1.03$ and $V_2'/V_2 = 2100/2000 = 1.05$. So, $V_1 > V_2$ and $V_1'/V_1 < V_2'/V_2$, i.e. this case is different from the one of relations (1), (5). At the same time, we have $V_1' - V_1 > V_2' - V_2$, $x_1' < x_1$ and $x_2' > x_2$ and according to Definition 2 a PPa occurs. It should also be mentioned that, because of $V_1'/V_1 < V_2'/V_2$, $x_1' < x_1$ and $x_2' > x_2$, the PPr doesn't occur (see Definition 1).

3.2 Population influence power absolute deviation as criterion

The major shortcoming of the approach based on relation (4) is that the total number of inhabitants in the two consecutive apportionments, V and V', is usually different, while the number of seats is the same

$(M = M')$. Under such conditions, the power of influence of one inhabitant [8] in the two apportionments, r and r', is different: $r = M/V$ and $r' = M/V'$. Therefore, the use of the number of inhabitants' absolute deviation $(V_i' - V_i)$ as a criterion is not correct. For each state, the comparison should not be based on the number V_i of inhabitants, but on the legal power of influence of the decisions of the House of Representatives delegated by the V_i inhabitants [8] $R_i = rV_i = MV_i/V = V_i/Q$, $i = \overline{1, n}$, known also as standard quota. For these reasons, at $c = V'/V$, further on there are formulated and characterized cases (a), (b) and (c) -– claimants in defining the population paradox; at a first glance, the population paradox would occur if:

a) at $R_i' \geq R_i$, relation $x_i' < x_i$ would also occur;

b) at $R_k' - R_k > R_j' - R_j$, relations (2) and (3) would also occur;

c) at $R_k' - x_k > R_j' - x_j$, relations (2) and (3) would also occur.

Case (a) outlines the conditions, for a particular state (i) within the two apportionments, needed for the population paradox to take place. In the other two cases, (b) and (c), the identification of population paradox is based on comparing the characteristics of two states; at the same time, the respective relations also contain, as further on will be ascertained, a parameter ($c = V'/V$ or $Q' = V'/M$) that refers to the entire apportionment.

The first, out of the two conditions of case (a), reflects the following situation: if the legal power of influence of state i (R_i), delegated by its population (V_i), does not decrease, then the power of influence of this state on the House of Representatives decisions, determined by the number of seats allocated to it (x_i), should not decrease as well. This condition can also be represented in another form. We have $R_i' \geq R_i$, that is $MV_i'/V' \geq MV_i/V$, implying $V_i'/V' \geq V_i/V$ or $V_i' \geq cV_i$.

The first condition of case (b) refers to the following situation: if the increase of the legal power of influence of state k, delegated by its population in two consecutive apportionments, is greater than the one of state j, then no seats should be taken from state k to be reallocated to state j. This condition can also be represented in another form. We have $R'_k - R_k > R'_j - R_j$, that is $MV'_k/V' - MV_k/V > MV'_j/V' - MV_j/V$, implying $VV'_k - V'V_k > VV'_j - V'V_j$ or $VV'_k - VV'_j > V'V_k - V'V_j$, therefore $V'_k - V'_j > c(V_k - V_j)$.

Finally, the essence of the first condition of case (c): if the increase of the legal power of influence of state k in the House of Representatives, determined by the x_k seats assigned to it according to the first apportionment, is greater than the one of state j, then no seats should be taken from state k to be reallocated to state j. This condition can also be represented in another form. We have $R'_k - x_k > R'_j - x_j$, that is $MV'_k/V' - x_k > MV'_j/V' - x_j$, implying $(V'_k - V'_j)/Q' > x_k - x_j$ or $V'_k - V'_j > Q'(x_k - x_j)$.

Let's consider these three cases. When comparing the two apportionments, there can be two approaches:

1) only states whose number of seats has been modified are taken into account;

2) all states, including those whose number of seats has not changed, are taken into account.

In this paper, approach 1 (the traditional, well known one, also used for PPr) is applied.

The advantage of case (a), of the (a) – (c) described above, is that the paradoxical situation is found in the entire apportionment, independently of any particular state. But this case not always specifies a paradox. For example, it may occur that a seat from state k is taken over by a state j, the power of influence of which has increased more than the power of influence of state k. Thus, case (a) would be a paradox only if a seat from state k was taken over by a state j, the power of influence of which has increased less than the one of state k, that is, only if case (b) occurs. So, case (a) would be a paradox only if case (b) would take place; given this, it is excluded.

For this reason, only cases (b) and (c) remain. Out of these, only case (b) fully corresponds to the requirements of population paradox for reasons described further on. As a basis of comparison regarding the first apportionment, the legal power of influence of each state delegated by its population, and not the one, determined by the number of seats allocated to state in the first apportionment, should be used. That is, condition $R'_k - R_k > R'_j - R_j$ should be used and not the $R'_k - x_k > R'_j - x_j$ one. The use of x_i instead of R_i, if $V_i \neq a_i Q$ (which usually takes place), favors (at $x_i = a_i + 1$) or disfavors (at $x_i = a_i$) state i. Also, when using PPr (see Definition 1), the comparison is made with V_i and not with x_i.

In case (b), in order to distinguish it from the already broadly used term "Population paradox" (see Definition 1) and also from the proposed new term for the last "Paradox of population rate" (PPr), in this paper the term "paradox of population influence absolute deviation" or, shorter, "Paradox of population influence" (PPi) will be used.

Thus, for PPi we have the definition below.

Definition 3. *The Paradox of population influence occurs, if at $M' = M$, $n' = n$ and*

$$V'_k - V'_j > V'(V_k - V_j)/V \tag{8}$$

the relations (2) and (3) take place.

3.3 Essential comparison of PPr, PPa and PPi approaches

The comparison by essence of PPr, PPa and PPi approaches regarding the population paradox can be made based on Definitions 1–3.

Regarding Definition 1 and Definition 2, the latter, as an approach, is closer to reflecting the conditions of population paradox manifestation, because it takes into account the absolute deviation of the number of votes $(V'_i - V_i)$, which is more appropriate for comparing the increase/decrease of the number of mandates from one poll to the next $(x'_i - x_i)$ than the ratio V'_i/V_i.

On the other hand, as it is shown in Section 3.2, the deviation $V'_i - V_i$ does not take into account the fact that the power of influence of an inhabitant in the two apportionments, r and r', differs. On the contrary, the relation (1), which is equivalent to the $R'_k/R_k >$

R'_j/R_j one, takes this fact into account. Indeed: we have $R_i = rV_i = MV_i/V$, implying $V_i = VR_i/M$, and similarly, $V'_i = R'_iV'/M$. So, $V'_k/V_k = R'_kV'/R_kV = cR'_k/R_k$ and, respectively, $V'_j/V_j = cR'_j/R_j$, Q.E.D. Therefore, from this point of view, Definition 1 better reflects the conditions of population paradox manifestation.

Thus, as regards Definitions 1 and 2, in one aspect (the power of influence of an inhabitant) – the PPr approach (Definition 1) is better, but in another aspect (the absolute deviation) – the PPa approach (Definition 2) is better. Definition 3 (Paradox of population influence – PPi) covers both these aspects; by comparison, this one uses pairs of states, and not separate states, and also for each distinct state i:

1) not the number of inhabitants V_i, but the power of influence of the elective body decisions R_i, delegated by the population V_i;

2) not the ratio V'_i/V_i, but, given the reasons mentioned in Section 3.2, the absolute deviation $D'_i - D_i$, suitable for comparing the increase/decrease of the number of seats x_i from one apportionment to the next one.

So, Definition 3 reflects more appropriately the population paradox manifestation and therefore the PPi approach is the only one that should be used for this purpose.

Statement 3. *At $V = V'$, the conditions of Absolute population paradox (Definition 2, PPa) and those of Paradox of population influence (Definition 3, PPi) coincide.*

Indeed, from (8), taking into account that $V = V'$, we have $V'_k - V'_j > V'(V_k - V_j)/V = V_k - V_j$ or $V'_k - V_k > V'_i - V_i$, that coincides with the (4) one. ■

4 Immunity of Hamilton method to PPr, PPa and PPi

For the population paradox within the meaning of Definition 3 (Paradox of population influence — PPi), Statement 4 takes place.

Statement 4. *Hamilton method is immune to the Paradox of population influence.*

Indeed, let's consider that condition (8) occurs. Taking into account that $Q' = cQ$ and $V_i = a_iQ + \Delta V_i$, where $a_i = \lfloor V_i/Q \rfloor$, relation (8) takes the form

$$a'_kQ' + \Delta V'_k - (a'_jQ' + \Delta V'_j) > c[a_kQ + \Delta V_k - (a_jQ + \Delta V_j)]$$

or

$$Q'[a'_k - a'_j - (a_k - a_j)] > c(\Delta V_k - \Delta V_j) + \Delta V'_j - \Delta V'_k. \qquad (9)$$

Obviously, the easiest case for state j to take over a seat from state k is: $x_k = a_k+1$, $x_j = a_j$, $x'_k = a'_k = a_k$ and $x'_j = a'_j+1 = a_j+1$, which can only be if $\Delta V_k > \Delta V_j$ and $\Delta V'_j > \Delta V'_k$. Thus, considering that $c > 0$, we have $c(\Delta V_k-\Delta V_j)+\Delta V'_j-\Delta V'_k > 0$ and $Q'[a'_k-a'_j-(a_k-a_j)] = 0$, that is condition (9) doesn't occur. Thus, conditions (2), (3) and (9) cannot occur simultaneously and, respectively, neither do conditions (2), (3) and (8). ■

In the context of Statement 1, let's examine two known examples of non-immunity of Hamilton method to the PPr, taken from [9, 10].

Example 4. *Let* $M = 11$, $n = 3$, $V = 1000$ *and* $V' = 1100$. *Other initial data taken from [9] and the results of calculations using Hamilton method are shown in Table 5.*

Table 5. Results of calculations to Example 4

State	V_i	V'_i	x_i	x'_i	V'_i/V_i	V'_i-V_i	R'_i-R_i	PPr/PPa/PPi
A	54	56	0	1	1.037	2	-0.034	Yes/Yes/No
B	243	255	3	2	1.049	12	-0.123	
C	703	789	8	8	1.122	86	0.157	

Data of Table 5 show that for states A and B the PPr and PPa paradoxes occur, but the PPi one doesn't. Even though the population of states A and B increased, and that of B increased more than that of A, their influence power (R_A and R_B) decreased and R_B decreased more (-0.123) than R_A (-0.034). So, the loss of a seat by state B to state A is not a paradox.

Example 5. *Let* $M = 10$*,* $n = 3$*,* $V = 10000$ *and* $V' = 9500$*. Other initial data taken from [10] and the results of calculations using Hamilton method are systemized in Table 6.*

Table 6. Results of calculations to Example 5

State	V_i	V_i'	x_i	x_i'	V_i'/V_i	$V_i' - V_i$	$R_i' - R_i$	PPr/PPa/PPi
1	1450	1470	2	1	1.014	20	0.097	Nes/Yes/Yo
2	3400	3380	3	4	0.994	-20	0.158	
3	5150	4650	5	5	0.903	-500	-0.255	

Data of Table 6 show that for states A and B the PPr and PPa paradoxes occurs, but the PPi one doesn't. Even though the population of state A increased and that of state B decreased, their influence power (R_A and R_B) increased, and R_B increased more (0.158) than R_A (0.097). So, the loss of a seat by state A to state B is not a paradox.

Although it is not the PPr, but the PPi that adequately portrays the situations of population paradox (and the Hamilton method is immune to PPi), the frequency of non-immunity of the Hamilton method to PPr and PPa is of a certain interest.

For this purpose, the SIMAP application for computer simulation has been specially developed and used.

The total number of votes for the second poll V' is determined as $V' = cV$. The values V_i', $i = 1, 2, \ldots, n$ for the second poll are random sizes determined as $V_i' = p_i V_i$, $i = 1, 2, \ldots, n$, with corrections required to make $V' = V_1' + V_2' + \ldots + V_n'$, where p_i is a stochastic size of uniform distribution in the range $[(c-1)(1-d); (c-1)(1+d)]$, $c < 2$, and d is a constant. For each variant of initial data (M, n, p and d) here and further on there was used a sample of 1 million random alternatives.

The character of percentages P_{Pr} and P_{Pa} dependence on n and d at $M = 101$ and $p = 0.02$ for the Hamilton method ($P_{Pr}(H)$ and $P_{Pa}(H)$) can be seen in Figure 1 and Figure 2. For $6 \leq M \leq 501$,

$3 \leq n \leq 50$, $0.02 \leq p \leq 0.1$ and $0.1 \leq d \leq 1$, $n < M$, the relations $0.018\% \leq P_{Pr}(H) \leq 4.66\%$ and $0.077\% \leq P_{Pa}(H) \leq 78.58\%$ occur.

In all cases, for the same values of initial data (M, n, p and d), the relation $P_{Pr}(H) < P_{Pa}(H)$ occurs; also the difference $P_{Pa}(H) - P_{Pr}(H)$, at $6 \leq M \leq 501$, $2 \leq n \leq 50$, $0.02 \leq p \leq 0.1$ and $0.1 \leq d \leq 1$, $n < M$, is increasing with the increase of M, p and d and, in most cases, with the increase of n, but the ratio $P_{Pa}(H)/P_{Pr}(H)$ is decreasing on n and d (Figure 3).

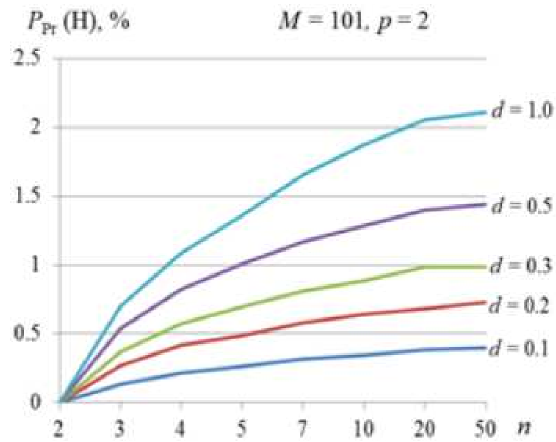

Figure 1. Dependence of P_{Pr} on n and d for Hamilton method

5 The non-immunity of d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë methods to the Paradox of population influence

The well-known d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë methods [2, 3, 11] are not immune to the PPi paradox. Examples 6–9 for each of the four methods are done below.

Example 6. *Let* $M = 101$*,* $n = 5$*,* $V = 10640$ *and* $V' = 10562$*. Other*

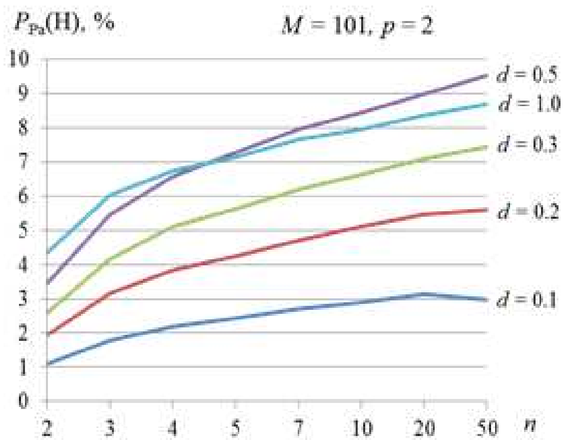

Figure 2. Dependence of P_{Pa} on n and d for Hamilton method

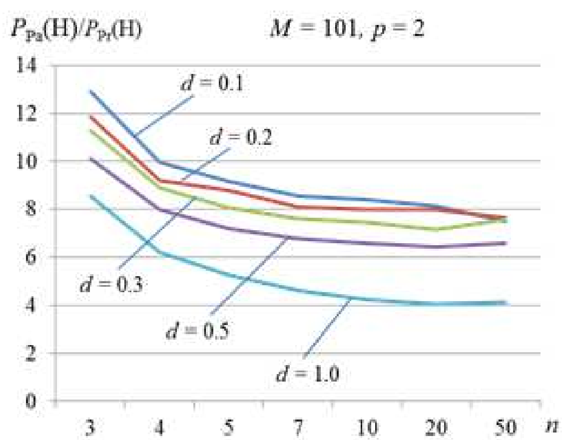

Figure 3. Dependence of P_{Pa}/P_{Pr} on n and d for Hamilton method

initial data and the results of calculations using the d'Hondt method are systemized in Table 7.

Table 7. Results of calculations to Example 6

State	V_i	V_i'	x_i	x_i'	V_i'/V_i	$V_i'-V_i$	$R_i'-R_i$	PPr/PPa/PPi
A	9900	9800	94	95	0.990	-100	-0.262	No/Yes/Yes
B	210	220	2	2	1.048	10	-0.110	
C	210	220	2	2	1.048	10	-0.110	
D	210	220	2	2	1.048	10	-0.110	
E	110	102	1	0	0.927	-8	-0.069	

Data of Table 7 show that for states A and E the PPi paradox occurs. Even though the power of influence of states A and E (R_A and R_E) decreased, and notwithstanding the fact that the power of influence of state A decreased more than that of state E ($R_E - R_E' = 0.069 < 0.262 = R_A - R_A'$), state E lost a seat in favor of state A. To be mentioned that, for this particular example, a similar situation occurs in relation with criterion $V_i' - V_i$ (see Table 7).

Example 7. *Let* $M = 101$*,* $n = 5$*,* $V = 8800$ *and* $V' = 8873$*. Other initial data and the results of calculations using Sainte-Laguë method are systemized in Table 8.*

Table 8. Results of calculations to Example 7

State	V_i	V_i'	x_i	x_i'	V_i'/V_i	$V_i'-V_i$	$R_i'-R_i$	PPr/PPa/PPi
A	8300	8230	94	95	0.992	-70	-1.581	No/Yes/Yes
B	150	200	2	2	1.333	50	0.555	
C	150	200	2	2	1.333	50	0.555	
D	150	200	2	2	1.333	50	0.555	
E	50	43	1	0	0.860	-7	-0.084	

Data of Table 8 show that for states A and E the PPi paradox occurs. Even though the influence power (R_A and R_E) of states A and E decreased, and those of state A decreased more than those of state E ($R_E - R'_E = 0.084 < 1.581 = R_A - R'_A$), state E lost a seat to state A. To mention that for this example a similar situation is with criterion $V'_i - V_i$ (see Table 8).

Example 8. *Let* $M = 101$*,* $n = 5$*,* $V = 8675$ *and* $V' = 8720$*. Other initial data and the results of calculations using Huntington-Hill method are systemized in Table 9.*

Table 9. Results of calculations to Example 8

State	V_i	V'_i	x_i	x'_i	V'_i/V_i	$V'_i - V_i$	$R'_i - R_i$	PPr/PPa/PPi
A	8130	8120	93	94	0.999	-10	-0.604	No/Yes/Yes
B	140	160	2	2	1.143	20	0.223	
C	140	160	2	2	1.143	20	0.223	
D	140	160	2	2	1.143	20	0.223	
E	125	120	2	1	0.960	-5	-0.065	

Data of Table 9 show that for states A and E the PPi paradox occurs. Even though the influence power (R_A and R_E) of states A and E decreased, and those of state A decreased more than those of state E ($R_E - R'_E = 0.065 < 0.604 = R_A - R'_A$), state E lost a seat to state A. To mention that for this example a similar situation is with criterion $V'_i - V_i$ (see Table 9).

Example 9. *Let* $M = 101$*,* $n = 5$*,* $V = 8685$ *and* $V' = 8755$*. Other initial data and the results of calculations using Adapted Sainte-Laguë method are systemized in Table 10.*

Data of Table 10 show that for states A and E the PPi paradox occurs. Even the influence power (R_A and R_E) of states A and E decreased, and those of state A decreased more than those of state E ($R_E - R'_E = 0.128 < 2.256 = R_A - R'_A$), state E lost a seat to state A.

Table 10. Results of calculations to Example 9

State	V_i	V_i'	x_i	x_i'	V_i'/V_i	$V_i' - V_i$	$R_i' - R_i$	PPr/PPa/PPi
A	8130	8000	93	94	0.984	-130	-2.256	No/Yes/Yes
B	140	210	2	2	1.5	70	0.795	
C	140	210	2	2	1.5	70	0.795	
D	140	210	2	2	1.5	70	0.795	
E	135	125	2	1	0.926	-10	-0.128	

To mention that for this example a similar situation is with criterion $V_i' - V_i$ (see Table 10).

It is of interest how often the d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë methods are not immune to the Paradox of population influence. With this aim, the SIMAP computer application was used, for the same initial data as for the Hamilton method in Section 4. The character of percentages P_{Pi} dependence on n and d for $M = 101$ and $p = 0.02$ for the d'Hondt ($P_{Pi}(dH)$), Sainte-Laguë ($P_{Pi}(SL)$), Huntington-Hill ($P_{Pi}(dH)$) and Adapted Sainte-Laguë ($P_{Pi}(ASL)$) methods can be seen in Figures 4–7.

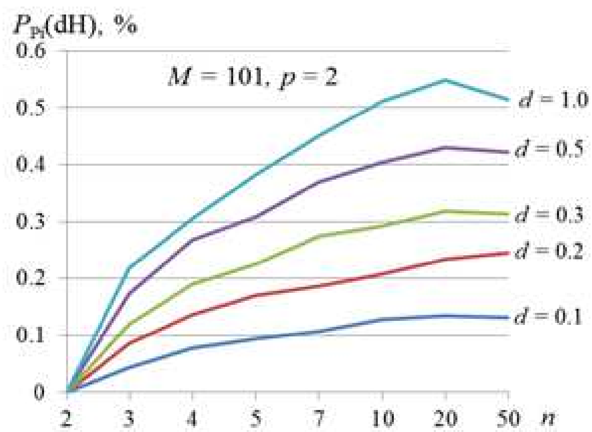

Figure 4. Dependence of P_{Pi} on n and d for d'Hondt method

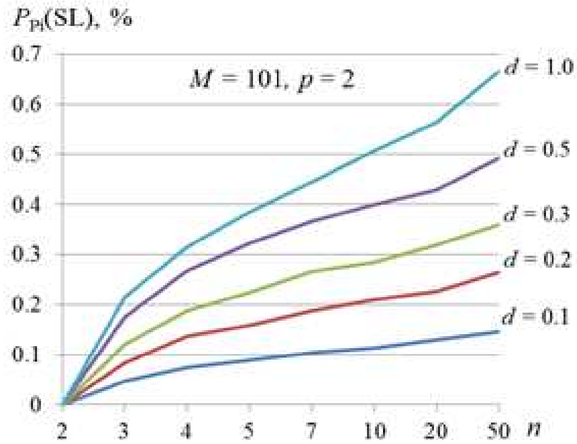

Figure 5. Dependence of P_{Pi} on n and d for Sainte-Laguë method

From Figures 4–7 one can see that, in terms of the percentage of non-immunity to the Paradox of population influence, there are no essential differences between the d'Hondt and Sainte-Laguë methods, nor between the Huntington-Hill and the Adapted Sainte-Laguë methods. Additional comparative data show that, from the point of view of immunity to PPi, in some cases the d'Hondt method is better than the Sainte-Laguë one and vice-versa; but the difference is not significant. The same situation is in the case of comparative analyses of Huntington-Hill and Adapted Sainte-Laguë methods.

Calculations performed show also that for $6 \leq M \leq 501$, $3 \leq n \leq 50$, $0.02 \leq p \leq 0.1$ and $0.1 \leq d \leq 1$, $n < M$, the following relations occur: $0.003\% \leq P_{Pi}(dH) \leq 0.652\%$ ($M = 501$, $n = 50$, $p = 0.1$, $d = 0.1$), $0.003\% \leq P_{Pi}(SL) \leq 0.806\%$ ($M = 101$, $n = 50$, $p = 0.1$, $d = 1$), $0.001\% \leq P_{Pi}(HH) \leq 0.617\%$ ($M = 501$, $n = 50$, $p = 0.1$, $d = 0.1$) and $0.001\% \leq P_{Pi}(ASL) \leq 0.621\%$ ($M = 501$, $n = 50$, $p = 0.1$, $d = 0.1$). So, for the specified range of initial data, the non-immunity to the Paradox of population influence, when using one of these four apportionment methods, does not exceed, on average, $0.6 - 0,8\%$, that is one case per 120-170 cases in total. For example, no one such case has been identified for the 11 apportionments of seats

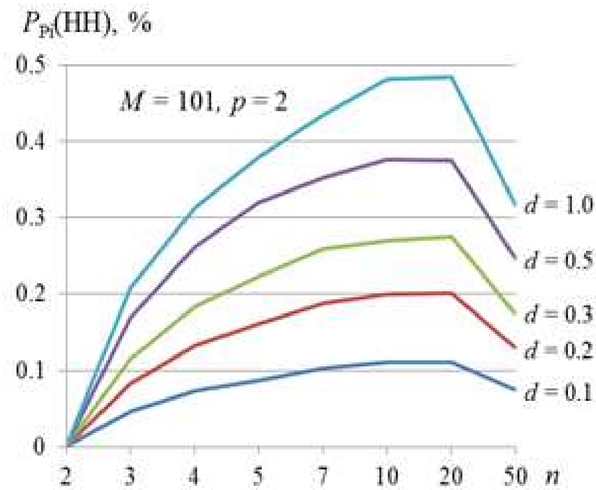

Figure 6. Dependence of P_{Pi} on n and d for H.-Hill method

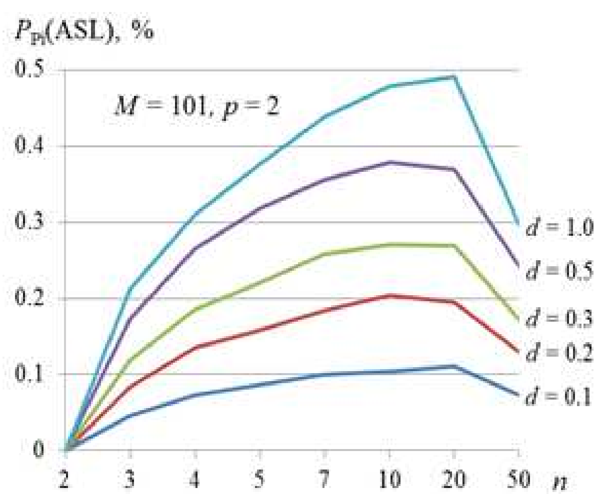

Figure 7. Dependence of P_{Pi} on n and d for ASL method

in the US Congress House of Representatives in 1900-2010 years, made applying the Webster (1900, 1910 and 1930 Census) and Huntington-Hill (1940-2010 Census) methods.

6 Conclusions

The well-known population paradox (PPr), based on the population deviation rate from one apportionment to the next one, is not always a true paradox. A new formulation of conditions of population paradoxical situations is proposed, which is based on the absolute deviation of the population power of influence. In order to distinguish it from the traditional one, for this particular case the term "Paradox of population influence" (PPi) is used. Of course, it would be better to use the term "population paradox", but in the new formulation. Thus, we count on the fact that the use of term PPi is temporary.

It is well known that the Hamilton method is not immune to the Population paradox (PPr), whereas the d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë divisor methods are [1–3, 11]. As for the Paradox of population influence (PPi), the situation is opposite: the Hamilton method is, and the d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë methods are not immune to it.

By computer simulation using the SIMAP application, the percentage of non-immunity of Hamilton method to PPr ($P_{Pr}(H)$) and PPa ($P_{Pa}(H)$), and of d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë methods to PPi ($P_{Pi}(dH)$, $P_{Pi}(SL)$), $P_{Pi}(HH)$ and $P_{Pi}(ASL)$, respectively) is estimated. It has been found that, for $6 \leq M \leq 501$, $3 \leq n \leq 50$, $0.02 \leq p \leq 0.1$ and $0.1 \leq d \leq 1$, $n < M$, the following relations occur:

- $0.018\% \leq P_{Pr}(H) \leq 4.66\%$;
- $0.077\% \leq P_{Pa}(H) \leq 78.58\%$;
- $0.003\% \leq P_{Pi}(dH) \leq 0.652\%$;

- $0.003\% \leq P_{Pi}(SL) \leq 0.806\%$;
- $0.001\% \leq P_{Pi}(HH) \leq 0.617\%$;
- $0.001\% \leq P_{Pi}(ASL) \leq 0.621\%$.

To mention that in all cases, for same values of initial data (M, n, p and d), the relation $P_{Pr}(H) < P_{Pa}(H)$ occurs. Also, for the specified range of initial data, the percentage of non-immunity to the Paradox of population influence, when using one of the four examined divisor methods (d'Hondt, Sainte-Laguë, Huntington-Hill and Adapted Sainte-Laguë), does not exceed, on average, 0.6-0,8%, that is one case per 120-170 cases in total.

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